# Chapter 2

# Stress, Strain and Isotropic Elasticity

## 2.7 Exercise Problems

**2.1** A strip of metal is subjected to a uniform tension stress $\sigma_{yy} = \sigma$, as shown in Fig. 2.16.

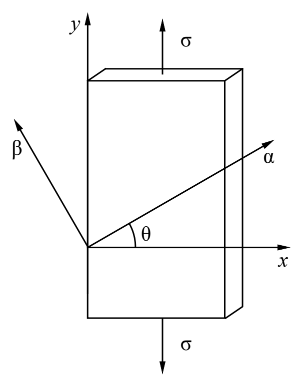



Figure 2.16: A strip of metal subjected to tensile stress.

(a) Calculate the shear stress $\sigma_{\alpha\beta}$ acting on a plane that is 45° to the horizontal axis.

(b) Write a general expression for the shear stress $\sigma_{\alpha\beta}$ on a plane making an arbitrary angle $\theta$ with the horizontal axis.

(c) Show that $\sigma_{\alpha\beta}$ is maximum at $\theta = 45°$.

(d) Calculate the hydrostatic pressure $p$ in the strip.

**Solution**
(a) $\sigma_{\alpha\beta} = (\cos 45°)(\sin 45°)\,\sigma_{yy} = \frac{1}{2}\,\sigma$.

(b) $\sigma_{\alpha\beta} = (\cos\theta)(\sin\theta)\,\sigma_{yy} = \frac{1}{2}\sin(2\theta)\,\sigma$.

(c) Maximum of $\sin(2\theta)$ occurs at $\theta = 45°$ where $\sin(2\theta) = \sin 90° = 1$. Therefore, maximum of $\sigma_{\alpha\beta}$ occurs at $\theta = 45°$ where $\sigma_{\alpha\beta} = \frac{1}{2}\,\sigma$.

(d) $p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -\frac{1}{3}\,\sigma$.

**2.2** An FCC single crystal thin film with the orientation shown in Fig. 2.17 is subjected to an equal-biaxial tensile stress $\sigma$. Plastic deformation of this crystal will occur by slip on the $(1\,\bar{1}\,1)$ plane (shaded) and along the $[0\,1\,1]$ direction as shown. Write an expression for the shear stress acting on the slip plane in the slip direction.

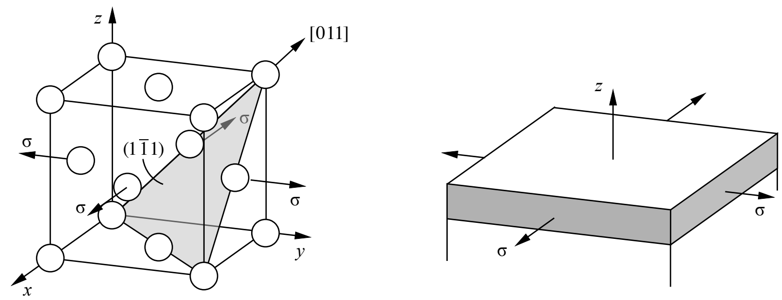



Figure 2.17: An FCC single crystal thin film under biaxial tension.

**Solution**
In $(x\,y\,z)$ coordinate system,

$$\sigma_{xx} = \sigma_{yy} = \sigma\,, \quad \sigma_{zz} = 0$$

Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the $[0\,1\,1]$, $[1\,\bar{1}\,1]$, and $[2\,1\,\bar{1}]$ directions respectively. The stress component in question is $\sigma_{x'y'}$.

$$\begin{aligned}\sigma_{x'y'} &= Q_{1k}\,Q_{2l}\,\sigma_{kl} \\ &= Q_{11}\,Q_{21}\,\sigma_{xx} + Q_{12}\,Q_{22}\,\sigma_{yy}\end{aligned}$$

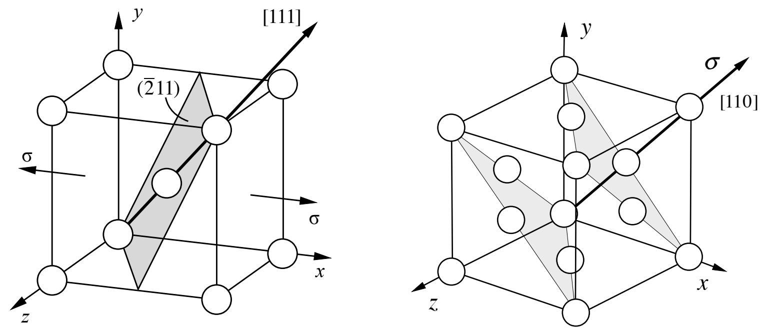
where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned}Q_{11} &= \frac{1}{\sqrt{2}}[0\,1\,1]\cdot[1\,0\,0] = 0 \\ Q_{21} &= \frac{1}{\sqrt{3}}[1\,\bar{1}\,1]\cdot[1\,0\,0] = \frac{1}{\sqrt{3}} \\ Q_{12} &= \frac{1}{\sqrt{2}}[0\,1\,1]\cdot[0\,1\,0] = \frac{1}{\sqrt{2}} \\ Q_{22} &= \frac{1}{\sqrt{3}}[1\,\bar{1}\,1]\cdot[0\,1\,0] = -\frac{1}{\sqrt{3}}\end{aligned}$$

Therefore,

$$\sigma_{x'y'} = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{3}}\right)\sigma = -\frac{1}{\sqrt{6}}\,\sigma$$

**2.3** A single crystal of BCC iron is pulled in uniaxial tension along a cube direction, $[1\,0\,0]$. We assume slip occurs on the $(\bar{2}\,1\,1)$ plane and in the $[1\,1\,1]$ direction as shown in Fig. 2.18(a). The critical resolved shear stress (CRSS) for slip to occur is 10 MPa. Calculate the tensile stress at which plastic yielding begins.




(a) (b)

Figure 2.18: (a) A single crystal of BCC iron subjected to tension along the $[1\,0\,0]$ direction. (b) An FCC single crystal subjected to tension along the $[1\,1\,0]$ direction. Two $(1\,1\,\bar{1})$ planes are shaded to help show the atomic positions.

**Solution**

In $(x\,y\,z)$ coordinate system,

$$\sigma_{xx} = \sigma\,, \quad \sigma_{zz} = \sigma_{yy} = 0$$

Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the $[1\,1\,1]$, $[\bar{2}\,1\,1]$, and $[0\,\bar{1}\,1]$ directions respectively. The stress component for the CRSS is $\sigma_{x'y'}$.

$$\begin{aligned} \sigma_{x'y'} &= Q_{1k}\,Q_{2l}\,\sigma_{kl} \\ &= Q_{11}\,Q_{21}\,\sigma_{xx} \end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned} Q_{11} &= \frac{1}{\sqrt{3}}[1\,1\,1]\cdot[1\,0\,0] = \frac{1}{\sqrt{3}} \\ Q_{21} &= \frac{1}{\sqrt{6}}[\bar{2}\,1\,1]\cdot[1\,0\,0] = -\frac{2}{\sqrt{6}} \end{aligned}$$

Therefore,

$$\sigma_{x'y'} = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{6}}\right)\sigma = -\frac{\sqrt{2}}{3}\,\sigma$$

$$|\sigma_{x'y'}| = 10\,\mathrm{MPa}\sigma \quad = \frac{3}{\sqrt{2}} \times 10\,\mathrm{MPa} = 21.2\,\mathrm{MPa}$$

**2.4** An FCC single crystal is subjected to a uniform tensile stress of magnitude $\sigma$ in the $[1\,1\,0]$ direction, as shown in Fig. 2.18(b). Calculate all the non-zero stress components in the cubic $(xyz)$ axes.

**Solution**

Let the $x$, $y$, $z$ axes in the $(x\,y\,z)$ coordinate system be aligned along the [1 0 0], [0 1 0], and [0 0 1] directions, respectively. Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the [1 1 0], $[\bar{1}\,1\,1]$, and $[1\,\bar{1}\,2]$ directions respectively. Here we know that

$$\sigma_{x'x'} = \sigma\,, \quad \sigma_{y'y'} = \sigma_{z'z'} = 0$$

The stress components in the $(x\,y\,z)$ coordinate system are

$$\begin{aligned} \sigma_{ij} &= Q^{\mathrm{T}}_{ik}\, Q^{\mathrm{T}}_{jl}\, \sigma'_{kl} \\ &= Q^{\mathrm{T}}_{i1}\, Q^{\mathrm{T}}_{j1}\, \sigma_{x'x'} \end{aligned}$$

where $Q^{\mathrm{T}}_{ik} = (\hat{e}'_k \cdot \hat{e}_i)$.

$$\begin{aligned} Q^{\mathrm{T}}_{11} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[1\,0\,0] = \frac{1}{\sqrt{2}} \\ Q^{\mathrm{T}}_{21} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[0\,1\,0] = \frac{1}{\sqrt{2}} \\ Q^{\mathrm{T}}_{31} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[0\,0\,1] = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{xx} &= Q^{\mathrm{T}}_{11}\, Q^{\mathrm{T}}_{11}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{yy} &= Q^{\mathrm{T}}_{21}\, Q^{\mathrm{T}}_{21}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{xy} &= Q^{\mathrm{T}}_{11}\, Q^{\mathrm{T}}_{21}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{xz} &= \sigma_{yz} = \sigma_{zz} = 0 \end{aligned}$$

**2.5** Consider a single crystal rod subjected to a uniaxial tensile stress $\sigma$, as shown in Fig. 2.19. Plastic deformation would occur if the resolved shear stress $\tau$ on a certain crystallographic plane (with normal vector $\hat{n}$) and along a certain crystallographic direction $\hat{m}$ exceeds a threshold value. Express the resolved shear stress $\tau$ in terms of the tensile stress $\sigma$ and the angles $\theta$ and $\phi$ that vectors $\hat{n}$ and $\hat{m}$ make with the tensile axis. The ratio $S = \tau/\sigma$ is called the *Schmid factor.*

**Solution**

$$\begin{aligned} \tau &= \cos\theta\,\cos\phi\;\sigma \\ S &= \cos\theta\,\cos\phi \end{aligned}$$

**2.6** The stress field of an infinite isotropic elastic medium containing a pressurized cylindrical hole (of radius $r_0$) has the following form,

$$\begin{aligned} \sigma_{rr} &= -\frac{A}{r^2} \\ \sigma_{\theta\theta} &= \frac{A}{r^2} \\ \sigma_{r\theta} &= 0 \end{aligned} \tag{2.81}$$

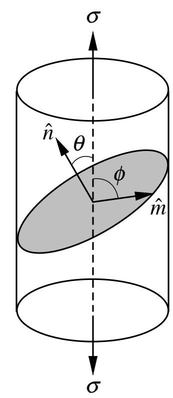



Figure 2.19: Single crystal subjected to uniaxial tension.

for $r \geq r_0$. Transform the stress field into Cartesian components: $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$. Make contour plots of $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$.

**Solution**

Let coordinate system A be the cylindrical $(r, \theta, z)$ coordinate system and coordinate system A' be the Cartesian $(x, y, z)$ coordinate system. Let matrix $Q$ be,

$$Q_{ik} = (\hat{e}_i^{xyz} \cdot \hat{e}_k^{r\theta z}) = \begin{bmatrix} (\hat{e}_x \cdot \hat{e}_r) & (\hat{e}_x \cdot \hat{e}_\theta) & (\hat{e}_x \cdot \hat{e}_z) \\ (\hat{e}_y \cdot \hat{e}_r) & (\hat{e}_y \cdot \hat{e}_\theta) & (\hat{e}_y \cdot \hat{e}_z) \\ (\hat{e}_z \cdot \hat{e}_r) & (\hat{e}_z \cdot \hat{e}_\theta) & (\hat{e}_z \cdot \hat{e}_z) \end{bmatrix}$$

Given that

$$\hat{e}_r = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta$$
$$\hat{e}_\theta = -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta$$

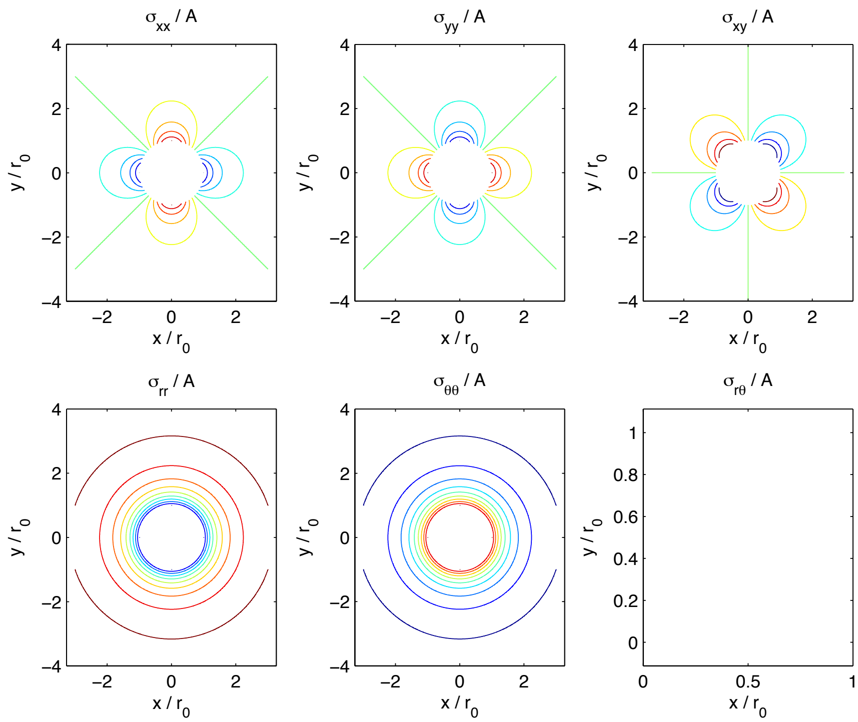
we have,

$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\sigma_{xx} &= Q_{xr}\, Q_{xr}\, \sigma_{rr} + Q_{x\theta}\, Q_{x\theta}\, \sigma_{\theta\theta} = -(\cos^2\theta - \sin^2\theta)\left(\frac{A}{r^2}\right) \\
\sigma_{yy} &= Q_{yr}\, Q_{yr}\, \sigma_{rr} + Q_{y\theta}\, Q_{y\theta}\, \sigma_{\theta\theta} = (\cos^2\theta - \sin^2\theta)\left(\frac{A}{r^2}\right) \\
\sigma_{zz} &= Q_{zr}\, Q_{zr}\, \sigma_{rr} + Q_{z\theta}\, Q_{z\theta}\, \sigma_{\theta\theta} = 0 \\
\sigma_{xy} &= Q_{xr}\, Q_{yr}\, \sigma_{rr} + Q_{x\theta}\, Q_{y\theta}\, \sigma_{\theta\theta} = -2\sin\theta\cos\theta\left(\frac{A}{r^2}\right) \\
\sigma_{yz} &= Q_{yr}\, Q_{zr}\, \sigma_{rr} + Q_{y\theta}\, Q_{z\theta}\, \sigma_{\theta\theta} = 0 \\
\sigma_{xz} &= Q_{xr}\, Q_{zr}\, \sigma_{rr} + Q_{x\theta}\, Q_{z\theta}\, \sigma_{\theta\theta} = 0
\end{aligned}$$

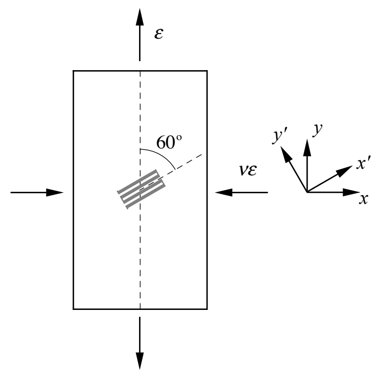
The results can be simplified as: $\sigma_{xx} = -A\cos(2\theta)/r^2$, $\sigma_{yy} = A\cos(2\theta)/r^2$, $\sigma_{xy} = -A\sin(2\theta)/r^2$. They can be written in terms of $x$, $y$, $z$ as

$$
\begin{aligned}
\sigma_{xx} &= -A\frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\sigma_{yy} &= \quad A\frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\sigma_{xy} &= -A\frac{2xy}{(x^2 + y^2)^2}
\end{aligned}
$$




**2.7** A strain gauge measures tensile strain along a given direction through the change of electrical resistance of thin wires aligned in that direction. Consider a solid subjected to uniaxial tensile stress, which induces a normal strain $\varepsilon$ in the tensile direction, and a normal strain $-\nu\varepsilon$ in the perpendicular directions, as shown in Fig. 2.20(a). Express the strain measured by a strain gauge oriented at 60° from the tensile axis.

**Solution**




For the coordinate system $(x\,y)$ shown,

$$\varepsilon_{xx} = -\nu\,\varepsilon$$
$$\varepsilon_{yy} = \varepsilon$$

The coordinate system $(x'\,y')$ is rotated from the coordinate system $(x\,y)$ by 30°. What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl}\\ &= Q_{11}\,Q_{11}\,\varepsilon_{xx} + Q_{12}\,Q_{12}\,\varepsilon_{yy}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

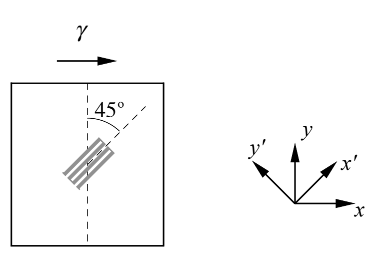
$$\begin{aligned}Q_{11} &= \cos 30^\circ = \frac{\sqrt{3}}{2}\\ Q_{12} &= \cos 60^\circ = \frac{1}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2}\,(-\nu\,\varepsilon) + \frac{1}{2}\,\frac{1}{2}\,\varepsilon\\ &= \frac{1-3\nu}{4}\,\varepsilon\end{aligned}$$

**2.8** Express the strain measured by a strain gauge oriented at 45° from the vertical axis of a solid subjected to an engineering shear strain of $\gamma$, as shown in Fig. 2.20(b).

**Solution**



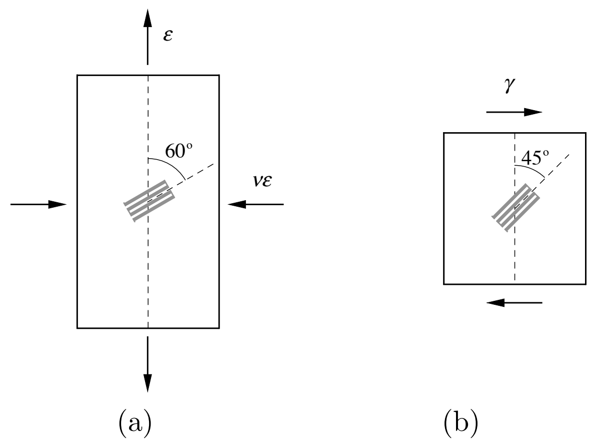



Figure 2.20: Strain gauge.

For the coordinate system $(x\,y)$ shown,

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\gamma}{2}$$

The coordinate system $(x'\,y')$ is rotated from the coordinate system $(x\,y)$ by 45°. What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl} \\ &= Q_{11}\,Q_{12}\,\varepsilon_{xy} + Q_{12}\,Q_{11}\,\varepsilon_{yx}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned}Q_{11} &= \cos 45^\circ = \frac{\sqrt{2}}{2} \\ Q_{12} &= \cos 45^\circ = \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\,\frac{\gamma}{2} + \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\,\frac{\gamma}{2} \\ &= \frac{\gamma}{2}\end{aligned}$$

**2.9** A thin film is subjected to strain $\varepsilon_{xx} = \varepsilon_1$, $\varepsilon_{yy} = \varepsilon_2$, $\varepsilon_{zz} = \varepsilon_3$. Write an expression for the axial strain $\varepsilon$ in the arbitrary direction shown in Fig. 2.21.

**Solution**

Choose a coordinate system $(x'\,y'\,z')$ such that the $x'$-axis is along the the arbitrary direction shown in Fig. 2.21.

$$\begin{aligned}\varepsilon = \varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl} \\ &= Q_{11}\,Q_{11}\,\varepsilon_{xx} + Q_{12}\,Q_{12}\,\varepsilon_{yy} + Q_{13}\,Q_{13}\,\varepsilon_{zz}\end{aligned}$$

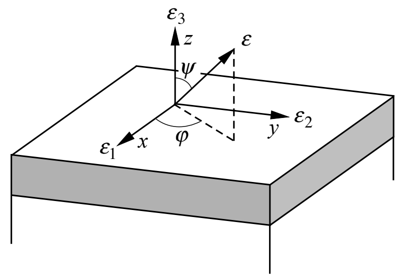



Figure 2.21: A thin film subjected to strain $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ in the $x$, $y$, $z$ directions, respectively.

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$
\begin{aligned}
Q_{11} &= \sin\psi\cos\varphi \\
Q_{12} &= \sin\psi\sin\varphi \\
Q_{13} &= \cos\psi
\end{aligned}
$$

Therefore,

$$\varepsilon = (\sin\psi\cos\varphi)^2\,\varepsilon_1 + (\sin\psi\sin\varphi)^2\,\varepsilon_2 + (\cos\psi)^2\,\varepsilon_3$$

**2.10** A solid cube with shear modulus $\mu$ and Poisson's ratio $\nu$ is subjected to a compressive stress along $z$, i.e. $\sigma_{zz} = -p$, but is not allowed to expand in the $x$ direction due to constraints imposed by rigid plates. The solid is allowed to expand freely in the $y$ direction. Obtain the normal stress and strain of the solid in $x$, $y$ and $z$ directions.

**Solution**

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]
\end{aligned}
$$

For this problem, $\varepsilon_{xx} = 0$ and $\sigma_{yy} = 0$. Therefore,

$$
\begin{aligned}
0 = \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{zz}\right) \\
\sigma_{xx} &= \nu\,\sigma_{zz} = -\nu\,p
\end{aligned}
$$

In summary,

$$\begin{aligned}
\sigma_{xx} &= -\nu\, p \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= -p \\
\epsilon_{xx} &= -\nu\, p \\
\epsilon_{yy} &= -\frac{\nu}{E}(\sigma_{xx}+\sigma_{zz}) = \frac{\nu(1+\nu)p}{E} \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz}-\nu\,\sigma_{xx}) = -\frac{(1-\nu^2)p}{E}
\end{aligned}$$

**2.11** Repeat the analysis in Problem 2.10 but for a solid constrained (i.e. unable to expand) in both $x$ and $y$ directions.

**Solution**

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx}-\nu(\sigma_{yy}+\sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy}-\nu(\sigma_{xx}+\sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu(\sigma_{xx}+\sigma_{yy})\right]
\end{aligned}$$

For this problem, $\varepsilon_{xx} = 0$ and $\varepsilon_{yy} = 0$. By symmetry, we have $\sigma_{xx} = \sigma_{yy}$. Therefore,

$$\begin{aligned}
0 = \varepsilon_{xx} &= \frac{1}{E}\left[(1-\nu)\sigma_{xx}-\nu\sigma_{zz}\right] \\
\sigma_{xx} = \sigma_{yy} &= \frac{\nu}{1-\nu}\,\sigma_{zz} = -\frac{\nu}{1-\nu}\,p
\end{aligned}$$

In summary,

$$\begin{aligned}
\sigma_{xx} &= -\frac{\nu}{1-\nu}\,p \\
\sigma_{yy} &= -\frac{\nu}{1-\nu}\,p \\
\sigma_{zz} &= -p \\
\epsilon_{xx} &= 0 \\
\epsilon_{yy} &= 0 \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz}-2\nu\,\sigma_{xx}) = -\frac{p}{E}\left(1-\frac{2\nu^2}{1-\nu}\right)
\end{aligned}$$

**2.12** Estimate the compressive stress that arises in a block of aluminum when the temperature increases from 0°C to 30°C if its volume is not allowed to expand. Use the thermoelastic properties of aluminum given in Table B.1. If the block were allowed to expand freely, it would experience a thermal strain of $\varepsilon^{\mathrm{T}} = \alpha\,\Delta T$ in all three direction with zero stress, where $\alpha$ is the thermal expansion coefficient and $\Delta T$ is the temperature increase.

**Solution**

$$\begin{aligned}
\varepsilon^{\rm tot}_{xx} &= \varepsilon^{\rm el}_{xx} + \varepsilon^{\rm T}_{xx} = \varepsilon^{\rm el}_{xx} + \alpha\Delta T = 0\\
\varepsilon^{\rm tot}_{yy} &= \varepsilon^{\rm el}_{yy} + \varepsilon^{\rm T}_{yy} = \varepsilon^{\rm el}_{yy} + \alpha\Delta T = 0\\
\varepsilon^{\rm tot}_{zz} &= \varepsilon^{\rm el}_{zz} + \varepsilon^{\rm T}_{zz} = \varepsilon^{\rm el}_{zz} + \alpha\Delta T = 0
\end{aligned}$$

Therefore, $\varepsilon^{\rm el}_{xx} = \varepsilon^{\rm el}_{yy} = \varepsilon^{\rm el}_{zz} = -\alpha\Delta T$.

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = B \cdot (\varepsilon^{\rm el}_{xx} + \varepsilon^{\rm el}_{yy} + \varepsilon^{\rm el}_{zz}) = -3B\,\alpha\,\Delta T$$

From Table B.1, we find $B = 75.9$ GPa, $\alpha = 23.1 \times 10^{-6}\,{\rm K}^{-1}$ for aluminum. Therefore, for $\Delta T = 30$ K, we have

$$\begin{aligned}
\sigma_{xx} = \sigma_{yy} = \sigma_{zz} &= -3B\,\alpha\,\Delta T\\
&= -3 \times 75.9 \times 23.1 \times 10^{-6} \times 30\,{\rm GPa}\\
&= 0.158\,{\rm GPa}\\
&= 158\,{\rm MPa}
\end{aligned}$$

**2.13** Consider an infinite elastic solid containing a cylindrical hole of radius $r_0$, which is filled with a gas with pressure $p$. The stress field in the solid is given in Eq. (2.81). Express the coefficient $A$ in terms of the pressure $p$ inside the hole and the hole radius $r_0$. Verify that this stress field satisfies the equilibrium condition, Eq. (2.56).

**Solution**

$$\begin{aligned}
-p = \sigma_{rr}|_{r=r_0} &= -\frac{A}{r_0^2}\\
A &= p\,r_0^2
\end{aligned}$$

Because $\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{rz} = 0$, and all the remaining stress components are independent of $\theta$ and $z$, it can be easily seen that, $(F_r = F_\theta = F_z = 0)$

$$\begin{aligned}
\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0\\
\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0
\end{aligned}$$

The only equilibrium condition we need to verify is

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Because $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$,

$$\begin{aligned}
\frac{\partial\sigma_{rr}}{\partial r} &= 2\,\frac{A}{r^3}\\
\frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) &= \frac{1}{r}\left(-\frac{A}{r^2} - \frac{A}{r^2}\right) = -2\,\frac{A}{r^3}
\end{aligned}$$

Therefore,

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Because $\sigma_{r\theta} = \sigma_{rz} = 0$, we have

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

**2.14** Given the stress field expression, Eq. (2.81), for an infinite elastic solid containing a pressurized cylindrical hole,

(a) Find the corresponding strain field. Express the strain field in Cartesian ($xyz$) components.

(b) Verify that the strain field satisfies the compatibility condition Eq. (2.67).

(c) Verify that the displacement field $u_r = C/r$, $u_\theta = 0$ is consistent with the strain field, and express the coefficient $C$ in terms of hole radius $r_0$ and pressure $p$.

Assume the solid is a linear isotropic material with with shear modulus $\mu$ and Poisson's ratio $\nu$.

**Solution**

(a) We have the plane strain condition $\varepsilon_{zz} = 0$. To find $\sigma_{zz}$, we use the condition that

$$\begin{aligned} 0 = \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] \\ \sigma_{zz} &= \nu(\sigma_{rr} + \sigma_{\theta\theta}) \end{aligned}$$

Given that $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$, $A = p\, r_0^2$ (see Problem 2.13), we have

$$\sigma_{zz} = 0$$

So we also have a plane stress condition in the solid (outside the hole).

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right] \\ &= -\frac{1+\nu}{E}\frac{A}{r^2} = -\frac{A}{2\mu}\frac{1}{r^2} \\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right] \\ &= \frac{1+\nu}{E}\frac{A}{r^2} = \frac{A}{2\mu}\frac{1}{r^2} \\ \varepsilon_{r\theta} &= \varepsilon_{rz} = \varepsilon_{\theta z} = 0 \end{aligned}$$

Similar to the stress transformation rule from cylindrical to Cartesian coordinates derived in Problem 2.6, we have similar expressions for the strain transformation, i.e.

$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
\varepsilon_{xx} &= Q_{xr}\,Q_{xr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{x\theta}\,\varepsilon_{\theta\theta} = -(\cos^2\theta - \sin^2\theta)\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{yy} &= Q_{yr}\,Q_{yr}\,\varepsilon_{rr} + Q_{y\theta}\,Q_{y\theta}\,\varepsilon_{\theta\theta} = (\cos^2\theta - \sin^2\theta)\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{zz} &= 0\\
\varepsilon_{xy} &= Q_{xr}\,Q_{yr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{y\theta}\,\varepsilon_{\theta\theta} = -2\sin\theta\cos\theta\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{yz} &= Q_{yr}\,Q_{zr}\,\varepsilon_{rr} + Q_{y\theta}\,Q_{z\theta}\,\varepsilon_{\theta\theta} = 0\\
\varepsilon_{xz} &= Q_{xr}\,Q_{zr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{z\theta}\,\varepsilon_{\theta\theta} = 0
\end{aligned}
$$

The results can be simplified as: $\sigma_{xx} = -A\cos(2\theta)/r^2$, $\sigma_{yy} = A\cos(2\theta)/r^2$, $\sigma_{xy} = -A\sin(2\theta)/r^2$. They can be written in terms of $x$, $y$, $z$ as

$$
\begin{aligned}
\varepsilon_{xx} &= -\frac{A}{2\mu}\,\frac{x^2-y^2}{(x^2+y^2)^2}\\
\varepsilon_{yy} &= \frac{A}{2\mu}\,\frac{x^2-y^2}{(x^2+y^2)^2}\\
\varepsilon_{xy} &= -\frac{A}{2\mu}\,\frac{2xy}{(x^2+y^2)^2}
\end{aligned}
$$

(b) The compatibility condition to be verified is

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

$$
\begin{aligned}
\varepsilon_{xx,yy} &= \frac{\partial^2}{\partial y^2}\varepsilon_{xx} = \frac{A}{2\mu}\,\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}\\
\varepsilon_{yy,xx} &= \frac{\partial^2}{\partial x^2}\varepsilon_{yy} = \frac{A}{2\mu}\,\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}\\
\varepsilon_{xy,xy} &= \frac{\partial^2}{\partial x\partial y}\varepsilon_{xy} = \frac{A}{2\mu}\,\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}
\end{aligned}
$$

Therefore,

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

(c) Suppose $u_r = C/r$ and $u_\theta = u_z = 0$, then

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = -\frac{C}{r^2} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{C}{r^2} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} = 0 \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) = 0 \\
\varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \\
\varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) = 0
\end{aligned}$$

These are consistent with the displacement fields in (a) as long as

$$C = \frac{A}{2\,\mu} = \frac{p\,r_0^2}{2\,\mu}$$

**2.15** Given the stress and strain expressions in Problems 2.13 and 2.14 for an infinite solid containing a pressurized cylindrical hole, express the total elastic energy per unit length along the hole in terms of the pressure $p$ and hole radius $r_0$ using the following two approaches and show that they agree with each other.

(a) Integrate the strain energy density $\Delta w$ over the entire volume of the solid.

(b) Integrate the work done on the interior surface of the cylindrical hole as the pressure gradually increases from 0 to $p$.

**Solution**
(a)

$$\begin{aligned}
\Delta w &= \frac{1}{2}\sigma_{rr}\varepsilon_{rr} + \frac{1}{2}\sigma_{\theta\theta}\varepsilon_{\theta\theta} = \frac{1}{2}\frac{A^2}{2\mu\,r^4} + \frac{1}{2}\frac{A^2}{2\mu\,r^4} \\
&= \frac{A^2}{2\mu\,r^4} \\
E &= \int_{r_0}^{\infty} r dr \int_0^{2\pi} d\theta\,\Delta w = \int_{r_0}^{\infty} dr\,2\pi\,r\,\frac{A^2}{2\mu\,r^4} \\
&= \frac{\pi\,A^2}{2\,\mu\,r_0^2} = \frac{\pi\,(p\,r_0^2)^2}{2\,\mu\,r_0^2} = \frac{\pi\,p^2\,r_0^2}{2\,\mu}
\end{aligned}$$

(b)

$$E = \frac{1}{2}\,p\,(2\pi\,r_0)\,u_r|_{r=r_0}$$

From Problem 2.14, we have $u_r = C/r$ with $C = p\,r_0^2/(2\mu)$, so that

$$u_r|_{r=r_0} = \frac{p\,r_0^2}{2\,\mu}\,\frac{1}{r_0}$$
$$E = \frac{1}{2}\,p\,(2\pi\,r_0)\left(\frac{p\,r_0^2}{2\,\mu}\,\frac{1}{r_0}\right) = \frac{\pi\,p^2\,r_0^2}{2\,\mu}$$

consistent with the result in (a).

**2.16** Find the expressions for the displacement field for Kelvin's problem, in which a unit force in the $z$ direction is applied at the origin in an infinite elastic medium. Express the displacement field in cylindrical coordinates. Which strain and stress components are non-zero in cylindrical coordinates?

**Solution**
In Cartesian coordinates, the displacement field caused by a point force in the $z$-direction is,

$$u_{\mathrm{x}} = \frac{1}{16\pi\mu(1-\nu)}\,\frac{x\,z}{(x^2+y^2+z^2)^{3/2}}$$
$$u_{\mathrm{y}} = \frac{1}{16\pi\mu(1-\nu)}\,\frac{y\,z}{(x^2+y^2+z^2)^{3/2}}$$
$$u_{\mathrm{z}} = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{3-4\nu}{(x^2+y^2+z^2)^{1/2}} + \frac{z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

In cylindrical coordinates $(r\,\theta\,z)$, where $r = \sqrt{x^2+y^2}$, the displacement field is,

$$u_r = \frac{1}{16\pi\mu(1-\nu)}\,\frac{r\,z}{(r^2+z^2)^{3/2}}$$
$$u_\theta = 0$$
$$u_z = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{3-4\nu}{(r^2+z^2)^{1/2}} + \frac{z^2}{(r^2+z^2)^{3/2}}\right]$$

In cylindrical coordinates, the strain field is related to the displacement field through,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$
$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$
$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$
$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

Given that $u_r$ and $u_z$ are independent of $\theta$ and $u_\theta = 0$, we have $\varepsilon_{r\theta} = \varepsilon_{\theta z} = 0$. The non-zero strain components are: $\varepsilon_{rr}$, $\varepsilon_{\theta\theta}$, $\varepsilon_{zz}$, and $\varepsilon_{rz}$.

From generalized Hooke's law, we have $\sigma_{r\theta} = \sigma_{\theta z} = 0$. The non-zero stress components are: $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, and $\sigma_{rz}$.