Chapter 2

Solutions to exercises

2.1. *Multiplication by an orthogonal matrix*
We are given an orthogonal matrix U.

(i) To prove that multiplication by an orthogonal matrix preserves lengths, we write

$$\|Ux\|^2 \;=\; (Ux)^\top (Ux) \;=\; x^\top U^\top U x \;\overset{(a)}{=}\; x^\top x \;=\; \|x\|^2, \tag{S2.1-1}$$

where (a) follows from (2.237).

(ii) To prove that multiplication by an orthogonal matrix preserves angles, we write

$$\langle Ux,\, Uy\rangle \;=\; (Ux)^\top (Uy) \;=\; x^\top U^\top U y \;\overset{(a)}{=}\; x^\top y \;=\; \langle x,\, y\rangle,$$

where (a) again follows from (2.237).

(iii) Let λ and v be an eigenvalue/eigenvector pair of U, that is, $Uv = \lambda v$. Then

$$\|v\| \;\overset{(a)}{=}\; \|Uv\| \;=\; |\lambda|\,\|v\|,$$

where (a) follows from (S2.1-1). Hence $|\lambda| = 1$ for any eigenvalue λ of U.

2.2. *Bases and frames in* $\mathbb{R}^2$

(i) The four matrices are

$$\Phi_1 \;=\; \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}, \qquad \Phi_2 \;=\; \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\Phi_3 \;=\; \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \Phi_4 \;=\; \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

(ii) Finding the dual basis or a dual frame is easiest using matrices. As long as each matrix above is of full rank (rank 2), we will be able to find the inverse (for bases/square matrices) or a right inverse (for frames/rectangular matrices),

$$\Phi\widetilde{\Phi}^\top \;=\; I.$$

To specifically find the canonical dual frame, use (2.160a),

$$\widetilde{\Phi} \;=\; (\Phi\Phi^*)^{-1}\,\Phi.$$

The synthesis operators for the four duals are

$$\widetilde{\Phi}_1 \;=\; \begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix}, \qquad \widetilde{\Phi}_2 \;=\; \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\widetilde{\Phi}_3 \;=\; \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \widetilde{\Phi}_4 \;=\; \begin{bmatrix} \frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{4} \end{bmatrix}.$$

The elements of the duals can be read off as columns of these matrices.

(iii) Φ_1 is a basis, and it is not orthonormal because it is not equal to its dual. (Alternatively, its two elements are not orthogonal.) Φ_2 is a frame, and it is tight because it is a scalar multiple of its canonical dual; it is furthermore 1-tight and equal to its canonical dual. (Alternatively, $\Phi_2\Phi_2^* = I$.) Φ_3 is a basis, and it is orthonormal because it is equal to its dual. (Alternatively, its two elements are orthogonal and have unit norm.) Φ_4 is a frame, and it is not tight because it is not a scalar multiple of its canonical dual. (Alternatively, $\Phi_4\Phi_4^* \neq cI_2$ for a scalar c.)

(iv) These projection coefficients can be computed as

$$\alpha_i \;=\; \widetilde{\Phi}_i^\top x,$$

and thus

$$\alpha_1 \;=\; \begin{bmatrix} 4 \\ -2\sqrt{3} \end{bmatrix}, \qquad \alpha_2 \;=\; \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{\sqrt{2}} \\ \sqrt{2} \\ 0 \end{bmatrix}, \qquad \alpha_3 \;=\; \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}, \qquad \alpha_4 \;=\; \begin{bmatrix} \frac{3}{2} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix}.$$

(v) We do this in more detail for the first representation than the others.

$$\begin{aligned}
\text{For } \Phi_1: \quad x \;&=\; \sum_{k=0}^{1} \alpha_{1,k}\varphi_{1,k} \;=\; \alpha_{1,0}\varphi_{1,0} + \alpha_{1,1}\varphi_{1,1} \\
&=\; 4\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - 2\sqrt{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ -2\sqrt{3} \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \\
\text{For } \Phi_2: \quad x \;&=\; \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix} - \frac{\sqrt{3}}{\sqrt{2}}\begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} + \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \\
\text{For } \Phi_3: \quad x \;&=\; 1\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} - \sqrt{3}\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \\
\text{For } \Phi_4: \quad x \;&=\; \frac{3}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix}.
\end{aligned}$$

(vi) We have already done this by computing duals in (ii).

(vii) The norm of x is $\|x\| = 2$. The norms of the expansion vectors are

$$\|\alpha_1\| \;=\; 2\sqrt{7}, \qquad \|\alpha_2\| \;=\; 2, \qquad \|\alpha_3\| \;=\; 2, \qquad \|\alpha_4\| \;=\; \sqrt{3}.$$

The orthonormal basis preserves the norm, as predicted by the Parseval equality (2.96). The tight frame also preserves the norm because it is a 1-tight frame (see (2.154)). The other two sets do not preserve the norm.

(viii) The expansions that produce more coefficients than the dimension of the signal are redundant. Thus, expansions with respect to Φ_2 and Φ_4 are redundant, while those with respect to Φ_1 and Φ_3 are not.

2.3. *Best approximation in* $\mathbb{R}^3$
The set $\{e_0,\ e_1,\ e_2\}$ forms an orthonormal basis in $\mathbb{R}^3$. Thus, the difference between the vector x and its approximation $\widehat{x}_{01}$ onto the (e_0, e_1)-plane is

$$\begin{aligned}
\|x - \widehat{x}_{01}\|^2 \;&=\; \|(\langle x,\, e_0\rangle - \alpha_0)e_0 + (\langle x,\, e_1\rangle - \alpha_1)e_1 + \langle x,\, e_2\rangle e_2\|^2 \\
&\overset{(a)}{=}\; |\langle x,\, e_0\rangle - \alpha_0|^2 + |\langle x,\, e_1\rangle - \alpha_1|^2 + |\langle x,\, e_2\rangle|^2,
\end{aligned}$$

where (a) follows from the Pythagorean theorem. This difference is the smallest possible and equal to $\langle x,\, e_2\rangle e_2$ when $\alpha_0 = \langle x,\, e_0\rangle$ and $\alpha_1 = \langle x,\, e_1\rangle$, that is, when $\widehat{x}_{01}$ is an orthogonal projection.

2.4. *Matrices representing bases and frames*

(i) We check norms and linear independence of the vectors in the set:

$$\begin{aligned}
\|\varphi_0\| \;=\; \|\varphi_1\| \;&=\; \|\varphi_2\| \;=\; 1, \\
\langle \varphi_0,\, \varphi_1\rangle \;&=\; \left(\sqrt{\frac{2}{3}}\right)\left(-\frac{1}{\sqrt{6}}\right) + \frac{1}{3} \;=\; 0, \\
\langle \varphi_0,\, \varphi_2\rangle \;&=\; \left(\sqrt{\frac{2}{3}}\right)\left(-\frac{1}{\sqrt{6}}\right) + \frac{1}{3} \;=\; 0, \\
\langle \varphi_1,\, \varphi_2\rangle \;&=\; \frac{1}{6} - \frac{1}{2} + \frac{1}{3} \;=\; 0.
\end{aligned}$$

Since all the φ vectors are orthogonal (and thus linearly independent), and there are exactly as many vectors as dimensions, Φ is a basis. Moreover, all φ are of unit norm, and thus Φ is represents an orthonormal basis.

(ii) The resulting matrix after projection is

$$\Phi' = \begin{bmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Since the matrix is of full rank 2, and there are 3 vectors, Φ' represents a frame. Moreover, since

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Φ' is also tight.

2.5. *Linear independence*

For U to be an independent set, it is necessary and sufficient that

$$\lambda_0 \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ ja & 1 \end{bmatrix} = \mathbf{0} \qquad \text{for } \lambda_0, \lambda_1, \lambda_2 \in \mathbb{C},$$

has the unique solution $\lambda_0 = \lambda_1 = \lambda_2 = 0$. This equation is equivalent to the system

$$\begin{aligned} a^2\lambda_0 + \lambda_1 &= 0, \\ \lambda_1 + ja\lambda_2 &= 0, \\ j\lambda_0 + (a-1)\lambda_1 + \lambda_2 &= 0. \end{aligned}$$

Thus, $a^2\lambda_0 = -\lambda_1 = ja\lambda_2$. By multiplying the last equation by a^2 and substituting for λ_0 and λ_1 from the first two, we get

$$\begin{aligned} ja^2\lambda_0 + (a-1)a^2\lambda_1 + a^2\lambda_2 &= j(ja\lambda_2) - (a-1)a^2(ja\lambda_2) + a^2\lambda_2 \\ &= (-a - j(a-1)a^3 + a^2)\lambda_2 \\ &= a(1 - ja^2)(a-1)\lambda_2 \\ &= a(1-ay)(1+ay)(a-1)\lambda_2 = 0, \end{aligned}$$

where $y = (1+j)/\sqrt{2}$ is the square root of j. Assuming that $a \notin \{0, 1, -1/y, 1/y\}$ means that $\lambda_2 = 0$. We also have $a^2\lambda_0 = -\lambda_1 = ja\lambda_2 = 0$ with $a \neq 0$ which ensures that $\lambda_0 = \lambda_1 = 0$. Hence, U is an independent set if and only if the complex number $a \notin \{0, 1, (1-j)/\sqrt{2}, -(1-j)/\sqrt{2}\}$.

For $a = j$, we see that

$$(-2)\begin{bmatrix} 0 & -1 \\ 0 & j \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & j-1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}.$$

2.6. *Continuity of the inner product*

We have

$$\begin{aligned} |\langle x+h_1,\, y+h_2\rangle - \langle x,\, y\rangle| &= |\langle x,\, h_2\rangle + \langle h_1,\, y\rangle + \langle h_1,\, h_2\rangle| \\ &\leq |\langle x,\, h_2\rangle| + |\langle h_1,\, y\rangle| + |\langle h_1,\, h_2\rangle| \\ &\overset{(a)}{\leq} \|x\|\,\|h_2\| + \|x\|\,\|h_2\| + \|h_1\|\,\|h_2\|, \end{aligned}$$

where (a) follows from the Cauchy–Schwarz inequality (2.29). Since the limit of the right-hand side is 0, so is the limit of the left-hand side, leading to the desired result.

2.7. *Inner product on $\mathbb{C}^N$*

Definition 2.7(i)–(ii) hold by the form of y^*Ax, regardless of any condition on A. For (iii) to hold,

$$x^*A^*y = (y^*Ax)^* = \langle x,\, y\rangle^* = \langle y,\, x\rangle = x^*Ay = x^*Ay,$$

implying $A = A^*$, that is, A must be a Hermitian operator. For (iv) to hold,

$$0 < \langle x,\, x\rangle = x^*Ax$$

for all nonzero x, which is precisely the definition of A being positive definite.

2.8. *Norms on* $\mathbb{C}^N$

To prove that v is a norm on a vector space, we use Definition 2.9.

For v_1:

(i) *Positive definiteness:* $v_1(x) = \sum_{k=0}^{N-1} |x_k|$ is always positive for any $x \in \mathbb{C}^N$ since it is a finite sum of positive terms ($|x_k| \geq 0$, for all $x_k \in \mathbb{C}$). Moreover,

$$\begin{aligned} v_1(x) = 0 \;&\Leftrightarrow\; \sum_{k=0}^{N-1} |x_k| = 0 \\ &\Leftrightarrow\; |x_k| = 0,\, k \in \{0,\, 1,\, \ldots,\, N-1\} \\ &\Leftrightarrow\; x_k = 0,\, k \in \{0,\, 1,\, \ldots,\, N-1\} \;\Leftrightarrow\; x = 0. \end{aligned}$$

(ii) *Positive scalability:*

$$v_1(\alpha x) \;=\; \sum_{k=0}^{N-1} |\alpha x_k| \;=\; \sum_{k=0}^{N-1} |\alpha|\,|x_k| \;=\; |\alpha| \sum_{k=0}^{N-1} |x_k| \;=\; |\alpha|\, v_1(x).$$

(iii) *Triangle inequality:*

$$\begin{aligned} v_1(x+y) \;&=\; \sum_{k=0}^{N-1} |x_k + y_k| \;\overset{(a)}{\leq}\; \sum_{k=0}^{N-1} (|x_k| + |y_k|) \\ &=\; \sum_{k=0}^{N-1} |x_k| + \sum_{k=0}^{N-1} |y_k| \;=\; v_1(x) + v_1(y), \end{aligned}$$

where (a) follows from the triangle inequality on $\mathbb{C}$.

For v_2:

(i) *Positive definiteness:* $v_2(x)$ is always positive for any $x \in \mathbb{C}^N$ since it is a the square root of a finite sum of positive terms ($|x_k|^2 \geq 0$, for all $x_k \in \mathbb{C}$). Moreover,

$$\begin{aligned} v_2(x) = 0 \;&\Leftrightarrow\; \left(\sum_{k=0}^{N-1} |x_k|^2\right) = 0 \\ &\Leftrightarrow\; |x_k|^2 = 0,\, k \in \{0,\, 1,\, \ldots,\, N-1\} \\ &\Leftrightarrow\; x_k = 0,\, k \in \{0,\, 1,\, \ldots,\, N-1\} \;\Leftrightarrow\; x = 0. \end{aligned}$$

(ii) *Positive scalability:*

$$v_2(\alpha x) \;=\; \left(\sum_{k=0}^{N-1} |\alpha x_k|^2\right)^{1/2} \;=\; \left(\sum_{k=0}^{N-1} |\alpha|^2\,|x_k|^2\right)^{1/2} \;=\; |\alpha|\, v_2(x).$$

(iii) *Triangle inequality:*

$$\begin{aligned} v_2(x+y) \;&=\; \left(\sum_{k=0}^{N-1} |x_k + y_k|^2\right)^{1/2} \;\overset{(a)}{\leq}\; \left(\sum_{k=0}^{N-1} |x_k|^2\right)^{1/2} + \left(\sum_{k=0}^{N-1} |y_k|^2\right)^{1/2} \\ &=\; v_2(x) + v_2(y), \end{aligned}$$

where (a) follows from Minkowski's inequality with $p = 2$.

2.9. *Norms on* $C([0,\, 1])$

To prove that v is a norm on a vector space, we use Definition 2.9.

For v_1:

(i) *Positive definiteness:* $v_1(x) = \int_0^1 |x(t)|\, dt$ is nonnegative for any $x \in V$ because the integrand is nonnegative. Moreover,

$$\begin{aligned} v_1(x) = 0 \;&\Leftrightarrow\; \int_0^1 |x(t)|\, dt = 0 \\ &\Leftrightarrow\; |x(t)| = 0,\; t \in [0,\, 1] \;\Leftrightarrow\; x = 0. \end{aligned}$$

(ii) *Positive scalability:*

$$v_1(\alpha x) \;=\; \int_0^1 |\alpha x(t)|\,dt \;=\; \int_0^1 |\alpha|\,|x(t)|\,dt \;=\; |\alpha| \int_0^1 |x(t)|\,dt \;=\; |\alpha|\,v_1(x).$$

(iii) *Triangle inequality:*

$$\begin{aligned} v_1(x+y) \;&=\; \int_0^1 |x(t)+y(t)|\,dt \;\le\; \int_0^1 (|x(t)|+|y(t)|)\,dt \\ &=\; \int_0^1 |x(t)|\,dt + \int_0^1 |y(t)|\,dt \;=\; v_1(x)+v_1(y). \end{aligned}$$

For v_2:

(i) *Positive definiteness:* $v_2^2(x) = \int_0^1 |x(t)|^2\,dt$ is nonnegative for any $x \in V$ because the integrand is nonnegative. Moreover,

$$\begin{aligned} v_2(x) = 0 \;&\Leftrightarrow\; \int_0^1 |x(t)|^2\,dt = 0 \\ &\Leftrightarrow\; |x(t)|^2 = 0,\; t \in [0,\,1] \;\Leftrightarrow\; x = 0. \end{aligned}$$

(ii) *Positive scalability:*

$$\begin{aligned} v_2(\alpha x) \;&=\; \left(\int_0^1 |\alpha x(t)|^2\,dt\right)^{1/2} \;=\; \left(\int_0^1 |\alpha|^2 |x(t)|^2\,dt\right)^{1/2} \\ &=\; |\alpha| \left(\int_0^1 |x(t)|^2\,dt\right)^{1/2} \;=\; |\alpha|\,v_2(x). \end{aligned}$$

(iii) *Triangle inequality:*

$$\begin{aligned} v_2(x+y) \;&=\; \left(\int_0^1 |x(t)+y(t)|^2\,dt\right)^{1/2} \\ &\overset{(a)}{\le}\; \left(\int_0^1 |x(t)|^2\,dt\right)^{1/2} + \left(\int_0^1 |y(t)|^2\,dt\right)^{1/2} \\ &=\; v_2(x)+v_2(y), \end{aligned}$$

where (a) follows from Minkowski's inequality with $p = 2$.

2.10. *Orthogonal transforms and ∞ norm*

(i) We can find the bounds a_2 and b_2 by considering rotations/rotoinversions of vectors on the unit circle. By the definition of the ∞-norm, the upper bound b_2 is clearly 1 since there is no vector on the unit circle whose maximum element is greater than 1. The lower bound is achieved when both components of the vector are equal, and thus

$$\frac{1}{\sqrt{2}} \;\le\; \|T_2 x\|_\infty \;\le\; 1.$$

(ii) By the same arguments we conclude that the upper bound is again 1 since there is no vector on the unit sphere whose maximum element is greater than 1. Similarly, the lower bound is achieved when all components of the vector are equal, and thus

$$\frac{1}{\sqrt{N}} \;\le\; \|T_N x\|_\infty \;\le\; 1.$$

2.11. *Cauchy–Schwarz inequality, triangle inequality, and parallelogram law*

(i) If one of the vectors is a zero vector, the result trivially holds. Suppose now that $\|x\| \neq 0$. Then, for any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} 0 \;&\overset{(a)}{\le}\; \langle \alpha x + y,\, \alpha x + y\rangle \;\overset{(b)}{=}\; \langle \alpha x,\, \alpha x + y\rangle + \langle y,\, \alpha x + y\rangle \\ &\overset{(c)}{=}\; \langle \alpha x,\, \alpha x\rangle + \langle \alpha x,\, y\rangle + \langle y,\, \alpha x\rangle + \langle y,\, y\rangle \\ &\overset{(d)}{=}\; |\alpha|^2\|x\|^2 + \alpha\langle x,\, y\rangle + \alpha^*\langle y,\, x\rangle + \|y\|^2, \end{aligned}$$

where (a) follows from the positive definiteness of the inner product; (b) and (c) from distributivity; and (d) from the linearity in the first argument and the conjugate-linearity in the second argument. Choosing $\alpha = -\langle y, x\rangle/\|x\|^2$, so $\alpha^* = \langle x, y\rangle/\|x\|^2$ and $|\alpha|^2 = |\langle x, y\rangle|^2/\|x\|^4$, we get

$$0 \le \frac{|\langle x, y\rangle|^2}{\|x\|^2} - 2\frac{|\langle x, y\rangle|^2}{\|x\|^2} + \|y\|^2 = -\frac{|\langle x, y\rangle|^2}{\|x\|^2} + \|y\|^2.$$

Multiplying through by $\|x\|^2$, rearranging, and taking the square root gives $|\langle x, y\rangle| \le \|x\|\,\|y\|$ as desired.

Note that the Cauchy–Schwarz inequality holds with equality if and only if inequality (a) above holds with equality. By positive definiteness of the inner product, that occurs if and only if $\alpha x + y = \mathbf{0}$, meaning that x is a scalar multiple of y.

(ii) We use the Cauchy–Schwarz inequality we just proved:

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\ &\overset{(a)}{\le} \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

where (a) follows from the Cauchy–Schwarz inequality, with equality if and only if x is a scalar multiple of y. Taking square roots gives the desired triangle inequality.

(iii) A simple proof is as follows:

$$\begin{aligned}&\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle \\ &= 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

(iv) Let the inner product be given by the real polarization identity (P2.11-1). We verify some properties of an inner product in Definition 2.7.

(i) *Distributivity:*

$$\begin{aligned}&\langle x+y, z\rangle \overset{(a)}{=} \frac{1}{4}\left(\|x+y+z\|^2 - \|x+y-z\|^2 \pm \|x+y+z\|^2\right) \\ &\overset{(b)}{=} \frac{1}{2}\|x+y+z\|^2 - \frac{1}{4}\left(\|x+y+z\|^2 + \|x+y-z\|^2\right) \\ &\overset{(c)}{=} \frac{1}{2}\|x+y+z\|^2 - \frac{1}{2}\left(\|x+y\|^2 + \|z\|^2\right) \\ &\overset{(d)}{=} \frac{1}{4}\left(\|x+y+2z\|^2 + \|x+y\|^2 - 2\|z\|^2\right) - \frac{1}{2}\left(\|x+y\|^2 + \|z\|^2\right) \\ &= \frac{1}{4}\|x+y+2z\|^2 - \frac{1}{4}\|x+y\|^2 - \|z\|^2 \\ &\overset{(e)}{=} \frac{1}{2}\|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \frac{1}{4}\|x-y\|^2 - \frac{1}{4}\|x+y\|^2 - \|z\|^2 \\ &\overset{(f)}{=} \frac{1}{2}\|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 - \|z\|^2 \\ &= \frac{1}{2}\|x+z\|^2 - \left(\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2\right) + \frac{1}{2}\|y+z\|^2 - \left(\frac{1}{2}\|y\|^2 + \frac{1}{2}\|z\|^2\right) \\ &\overset{(g)}{=} \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2\right) + \frac{1}{4}\left(\|y+z\|^2 - \|y-z\|^2\right) \\ &\overset{(h)}{=} \langle x, z\rangle + \langle y, z\rangle,\end{aligned}$$

where (a) follows from the polarization identity; (b) from adding and subtracting $\|x+y+z\|^2/4$; (c) from the parallelogram law applied to the summand in parentheses with $(x+y)$ and z as vectors; (d) from the parallelogram law applied to $\|x+y+z\|^2$ with $(x+y+z)$ and z as vectors; (e) from the parallelogram law applied to $\|x+y+2z\|^2$ with $(x+z)$ and $(y+z)$ as vectors;

(f) from the parallelogram law applied to the third and fourth summands; (g) from the parallelogram law applied to the terms in parentheses; and (h) from the polarization identity.

(ii) *Hermitian symmetry:* The arguments in the polarization identity commute in the real case, hence: $\langle y, x\rangle = \langle x, y\rangle$.

(iii) *Positive definiteness:*

$$\langle x, x\rangle \;=\; \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) \;=\; \frac{1}{4}\left(\|2x\|^2 - 0\right) \;=\; \|x\|^2,$$

so the positive definiteness of the inner product follows from the positive definiteness of the norm.

2.12. *Norm induced by an inner product*
To prove that v is a norm on a vector space, we use Definition 2.9.

(i) *Positive definiteness:* $v_2(x)$ is always positive since a square root is a nonnegative function. Moreover,

$$v(x) = 0 \;\Leftrightarrow\; \sqrt{\langle x, x\rangle} = 0 \;\Leftrightarrow\; x = 0.$$

(ii) *Positive scalability:*

$$v(\alpha x) \;=\; \sqrt{\langle \alpha x, \alpha x\rangle} \;=\; \sqrt{\alpha\alpha^* \langle x, x\rangle} \;=\; |\alpha|\sqrt{\langle x, x\rangle} \;=\; |\alpha|\, v(x).$$

(iii) *Triangle inequality:*

$$\begin{aligned} v^2(x+y) &= \langle x+y, x+y\rangle \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x+y, x\rangle^* + \langle x+y, y\rangle^* \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle x, y\rangle^* \\ &= v^2(x) + v^2(y) + 2\Re\{\langle x, y\rangle\}. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we further have that

$$\Re\{\langle x, y\rangle\} \;\leq\; |\langle x, y\rangle| \;\leq\; \sqrt{\langle x, x\rangle\langle y, y\rangle} \;=\; v(x)v(y).$$

Hence,

$$v^2(x+y) \;\leq\; v^2(x) + v^2(y) + 2v(x)v(y) \;=\; (v(x)+v(y))^2.$$

Taking the square root of both sides yields the desired property.

2.13. *Distances not necessarily induced by norms*
To show that such a discrete metric is a valid distance, we verify that it satisfies the four properties:

(i) *Nonnegativity:* For any $x, y \in V$, $d(x,y) \geq 0$.

(ii) *Symmetry:* For any $x, y \in V$, if $x \neq y$ then $d(x,y) = 1 = d(y,x)$; if $x = y$ then $d(x,y) = 0 = d(y,x)$.

(iii) *Triangle inequality:* For any $x, y, z \in V$ that are not all equal, $d(x,y) + d(y,z) \geq 1 \geq d(x,z)$. In case $x = y = z$, $d(x,y) + d(y,z) = 0 = d(x,z)$.

(iv) *Identity of Indiscernibles:* For any $x, y \in V$, by the definition of $d(x,y)$ we have $d(x,x) = 0$ and $d(x,y) = 0 \Rightarrow x = y$.

Hence, the defined discrete metric is a distance.

A simple example shows that $d(x,y)$ is not induced by any norm. Consider two vectors: $x = 2e_0$ has 2 in the first position and zeros elsewhere; $y = -2e_1$ has -2 in the second position and zeros elsewhere. Then, for any p,

$$\|x-y\|_p \;=\; (2^p + 2^p)^{1/p} = 2^{(p+1)/p} \;>\; 2 \;>\; 1,$$

while $d(x,y) = 1$.

2.14. *Convergence of the inner product in $\ell^2(\mathbb{Z})$*
Let x and y be sequences in $\ell^2(\mathbb{Z})$. Define x' and y' by

$$x'_n \;=\; |x_n| \qquad \text{and} \qquad y'_n \;=\; |y_n|, \qquad \text{for all } n \in \mathbb{Z}.$$

It is clear that $\|x'\| = \|x\|$ and $\|y'\| = \|y\|$, so x' and y' are in $\ell^2(\mathbb{Z})$. To check the absolute convergence of (2.22b), first note

$$\sum_{n\in\mathbb{Z}} |x_n\, y_n^*| \;=\; \sum_{n\in\mathbb{Z}} x'_n\, (y'_n)^*.$$

The latter sum is of nonnegative terms and thus must converge or diverge to ∞. However, divergence to infinity is not possible because the sum is bounded by $\|x'\|\,\|y'\|$ through the Cauchy–Schwarz inequality applied to $\langle x',\, y'\rangle$.

2.15. *Definition of ∞ norm*
Following the hint, it suffices to consider $x = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{N-1}\end{bmatrix}^\top$ with $|a_i| \le 1$ for $i = 1, 2, \ldots, N-1$. (Why? All the norms we are considering satisfy Definition 2.9(ii). Hence, multiplying by any scalar does not change whether the condition of interest holds. In addition, changing the order of elements in a vector does not change its norm.) We need to show that $\lim_{p\to\infty}\|x\|_p = 1$.

Because of the first entry of x, we have $\|x\|_p \ge 1$. We also have

$$\|x\|_p^p \;=\; 1 + a_1^p + a_2^p + \cdots + a_{N-1}^p \;\le\; N,$$

since $|a_i| \le 1$ for each i. Thus $\lim_{p\to\infty}\|x\|_p \le \lim_{p\to\infty} N^{1/p} = 1$. Combining the upper and lower bounds completes the proof.

2.16. *Quasinorms with $p < 1$*

(i) Let $x = \begin{bmatrix} 1 & 0\end{bmatrix}^\top$ and $y = \begin{bmatrix} 0 & 1\end{bmatrix}^\top$. Then

$$\|x+y\|_{1/2} \;=\; (1+1)^2 \;=\; 4 \;>\; 2 \;=\; 1+1 \;=\; \|x\|_{1/2} + \|y\|_{1/2},$$

violating Definition 2.9(iii).

(ii) Let $x \in \mathbb{R}^N$. $\|x\|_p^p$ is a sum of N terms: $\sum_{i=1}^N |x_i|^p$. Since a finite sum is always interchangeable with a limit, we have

$$\lim_{p\to 0}\|x\|_p^p \;=\; \sum_{i=0}^{N-1} \lim_{p\to 0}|x_i|^p.$$

In this sum, each nonzero x_i contributes 1 because $\lim_{p\to 0}|x_i|^p = 1$. Each zero x_i contributes 0. This proves that $\lim_{p\to 0}\|x\|_p^p$ gives the count of the number of nonzero components in x.

2.17. *Equivalence of norms on finite-dimensional spaces*

(i) (a) $\|x\|_1 \ge \|x\|_2$ because

$$\begin{aligned}
\|x\|_1^2 \;&=\; \left(\sum_{i=0}^{N-1} |x_i|\right)^2 \;=\; \sum_{i=0}^{N-1} |x_i|^2 + 2 \sum_{0\le i<j\le N-1} |x_i\, x_j| \\
&=\; \|x\|_2^2 + 2 \sum_{0\le i<j\le N-1} |x_i\, x_j| \;\ge\; \|x\|_2^2.
\end{aligned}$$

(b) $\|x\|_2 \ge \|x\|_\infty$ because

$$\|x\|_2^2 \;=\; \sum_{i=0}^{N-1} |x_i|^2 \;\ge\; \max_{i=0,2,\ldots,N-1} |x_i|^2 \;=\; \|x\|_\infty^2.$$

(c) $N\|x\|_\infty \ge \sqrt{N}\|x\|_2$ because

$$\begin{aligned}
(N\|x\|_\infty)^2 \;&=\; N^2 \max_{i=0,1,\ldots,N-1} |x_i|^2 \;=\; N\left(N \max_{i=0,1,\ldots,N-1} |x_i|^2\right) \\
&\ge\; N \sum_{i=0}^{N-1} |x_i|^2 \;=\; \left(\sqrt{N}\,\|x\|_2\right)^2.
\end{aligned}$$

(d) $\sqrt{N}\,\|x\|_2 \geq \|x\|_1$ because

$$\begin{aligned}\left(\sqrt{N}\,\|x\|_2\right)^2 - \|x\|_1^2 &= N\sum_{i=0}^{N-1}|x_i|^2 - \left(\sum_{i=0}^{N-1}|x_i|^2 + 2\sum_{0\leq i<j\leq N-1}|x_i\,x_j|\right)\\ &= (N-1)\sum_{i=0}^{N-1}|x_i|^2 - 2\sum_{0\leq i<j\leq N-1}|x_i\,x_j|\\ &= \sum_{0\leq i<j\leq N-1}(|x_i|-|x_j|)^2 \;\geq\; 0.\end{aligned}$$

(ii) If there exists $v \in V$ such that $\|v\|_a < \infty$ and $\|v\|_b = \infty$, then $\|\cdot\|_a$ and $\|\cdot\|_b$ are not equivalent. We give a counterexample in $\mathbb{C}^{\mathbb{Z}}$, the infinite-dimensional space of complex sequences.

(a) The sequence v with all $v_n = 1$ has a bounded ∞ norm, $\|v\|_\infty = 1$, and unbounded 1 norm and 2 norm. Hence the ∞ norm is not equivalent to the 1 norm and 2 norm on $\mathbb{C}^{\mathbb{Z}}$.

(b) The sequence v with

$$v_n \;=\; \begin{cases} 1/n, & n>0;\\ 0, & \text{otherwise,}\end{cases}$$

has an unbounded 1 norm

$$\begin{aligned}\|v\|_1 &= \left(\sum_{n=1}^{\infty}\frac{1}{n}\right)^{1/2} \;>\; \left(1+\sum_{n=2}^{\infty}2^{-\lceil\log_2 n\rceil}\right)^{1/2}\\ &= \left(1+\tfrac{1}{2}+\tfrac{1}{2}+\cdots\right)^{1/2} \;=\; \infty,\end{aligned}$$

and a bounded 2 norm

$$\|v\|_2 \;=\; \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2} \;=\; \frac{\pi}{\sqrt{6}}.$$

Hence the 1 norm is not equivalent to the 2 norm on $\mathbb{C}^{\mathbb{Z}}$.

2.18. *Nesting of ℓ^p spaces*

We prove the nesting property by induction.

(i) Let $x \in \ell^1(\mathbb{Z})$, so $\|x\|_1 < \infty$. Then

$$\|x\|_2^2 \;=\; \sum_{i\in\mathbb{Z}}|x_i|^2 \;\leq\; \left(\sum_{i\in\mathbb{Z}}|x_i|\right)^2 \;=\; \|x\|_1^2 \;<\; \infty.$$

Hence, $x \in \ell^2(\mathbb{Z})$, implying that $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(ii) Let $x \in \ell^p(\mathbb{Z})$, and for any $n = 1, 2, \ldots, p$ its n-norm $\|x\|_n < \infty$. Then

$$\|x\|_{p+1}^{p+1} \;=\; \sum_{i\in\mathbb{Z}}|x_i|^{p+1} \;\leq\; \sum_{i\in\mathbb{Z}}|x_i|^p\sum_{i\in\mathbb{Z}}|x_i| \;=\; \|x\|_p^p\,\|x\|_1 \;<\; \infty.$$

Hence, $x \in \ell^{p+1}(\mathbb{Z})$, implying that $\ell^p(\mathbb{Z}) \subset \ell^{p+1}(\mathbb{Z})$.

2.19. $\mathcal{L}^p([0,\,1])$ *spaces*

(i) We first show that the parallelogram law holds in $\mathcal{L}^2([0,\,1])$, which follows directly from the linearity of the integral. In fact,

$$\begin{aligned}\|x+y\|_2^2 + \|x-y\|_2^2 &= \int_0^1|x(t)+y(t)|^2\,dt + \int_0^1|x(t)-y(t)|^2\,dt\\ &= 2\left(\int_0^1|x(t)|^2\,dt + \int_0^1|y(t)|^2\,dt\right)\\ &= 2(\|x\|_2^2+\|y\|_2^2).\end{aligned}$$

(ii) We now show that the parallelogram law does not hold in $\mathcal{L}^p([0,\,1])$ for $p \neq 2$. Take $x(t) = t$ and $y(t) = 1-t$. Then

$$\begin{aligned}
\|x\|_p &= \left(\int_0^1 |t|^p\,dt\right)^{1/p} = \frac{1}{(p+1)^{1/p}},\\
\|y\|_p &= \left(\int_0^1 |1-t|^p\,dt\right)^{1/p} = \left(\int_0^1 |u|^p\,du\right)^{1/p} = \|x\|_p,\\
\|x-y\|_p &= \left(\int_0^1 |2t-1|^p\,dt\right)^{1/p} = \left(\frac{1}{2}\int_{-1}^1 |u|^p\,du\right)^{1/p} = \|x\|_p,\\
\|x+y\|_p &= \left(\int_0^1 |1|^p\,dt\right)^{1/p} = 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x+y\|_p^2 + \|x-y\|_p^2 &= 1 + \frac{1}{(p+1)^{2/p}} = \frac{(p+1)^{2/p}+1}{(p+1)^{2/p}},\\
2(\|x\|_p^2 + \|y\|_p^2) &= \frac{4}{(p+1)^{2/p}}.
\end{aligned}$$

The two are equal only when $p = 2$.

2.20. *Closed subspaces and* $\ell^0(\mathbb{Z})$

(i) For any $v \in \ell^0(\mathbb{Z})$, define $I = \{i \mid v_i \neq 0\}$ to be the finite set of indices of nonzero elements of v. Let $|I| = n < \infty$ be the number of such elements in v. Then,

$$\begin{aligned}
\|v\|_2 &= \left(\sum_{i\in\mathbb{Z}} |v_i|^2\right)^{1/2} = \left(\sum_{i\in I} |v_i|^2\right)^{1/2} \leq \left(n \max_{i\in I} |v_i|^2\right)^{1/2}\\
&= \sqrt{n}\max_{i\in I}|v_i| < \infty.
\end{aligned}$$

Hence, $v \in \ell^2(\mathbb{Z})$, which implies that $\ell^0(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(ii) Consider a sequence $v^{(n)} = \begin{bmatrix}\ldots & 0 & 1 & \frac{1}{2} & \ldots & \frac{1}{n} & 0 & \ldots\end{bmatrix} \in \ell^0(\mathbb{Z})$. Let also $\lim_{n\to\infty} v^{(n)} = v$. Since v has infinitely many nonzero elements, $v \notin \ell^0(\mathbb{Z})$. However, $v \in \ell^2(\mathbb{Z})$, since

$$\|v\|_2 = \left(\sum_{i\in\mathbb{Z}} |v_i|^2\right)^{1/2} = \left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right)^{1/2} = \frac{\pi}{\sqrt{6}} < \infty.$$

Hence, $\ell^0(\mathbb{Z})$ is not a closed subspace of $\ell^2(\mathbb{Z})$.

2.21. *Infinite sequences and completeness*

For the set $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be a basis for $\ell^2(\mathbb{Z})$, it would have to be possible to express any vector in $\ell^2(\mathbb{Z})$ uniquely with respect to the set. In this case, uniqueness is satisfied in that every vector in $\overline{\operatorname{span}}(\{\varphi_k\}_{k\in\mathbb{Z}})$ has a unique expansion with respect to the set. However, $\overline{\operatorname{span}}(\{\varphi_k\}_{k\in\mathbb{Z}})$ does not include every vector in $\ell^2(\mathbb{Z})$. For example, let

$$\psi_0 = \begin{bmatrix}\cdots & 0 & 0 & \boxed{\frac{1}{\sqrt{2}}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \cdots\end{bmatrix}^\top.$$

This vector is in $\ell^2(\mathbb{Z})$ but not in $\overline{\operatorname{span}}(\{\varphi_k\}_{k\in\mathbb{Z}})$. To see why it is not, note that the support of ψ_0 is $\{0, 1\}$, which overlaps with the support of only one vector in $\{\varphi_k\}_{k\in\mathbb{Z}}$, namely the sequence with no shift ($k = 0$). Since ψ_0 is not a scalar multiple of φ_0 and none of the shifted versions of φ_0 in the set has support overlapping with $\{0, 1\}$, there is no way to write ψ_0 as an expansion with respect to $\{\varphi_k\}_{k\in\mathbb{Z}}$.

(The choice of ψ_0 and its notation are suggestive. If we define ψ_k for nonzero $k \in \mathbb{Z}$ through

$$\psi_{k,n} = \psi_{0,n-2k}, \qquad n \in \mathbb{Z},$$

then $\{\varphi_k\}_{k\in\mathbb{Z}} \cup \{\psi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$.)

2.22. *Completeness*
We have that

$$\lim_{k\to\infty} p_k(t) \;=\; p(t),$$

where

$$p(t) \;=\; \frac{1}{1-\frac{1}{2}t}, \qquad 0 \le t \le 1.$$

While (p_k) is a Cauchy sequence in $\mathcal{P}$, it does not converge to a vector in $\mathcal{P}$ since $p(t)$ is not a polynomial. Therefore, $\mathcal{P}$ is an inner product space, but since it is not complete, it is not a Hilbert space.

2.23. *Completeness of* $\mathbb{C}^N$
A vector space is complete if every Cauchy sequence in it converges to an element of that vector space. Hence, given any Cauchy sequence $(v_n)_{n\ge0} \subset \mathbb{C}^N$, we must show that $\lim_{n\to\infty} v_n = v \in \mathbb{C}^N$.

Let $(v_n^{(i)})_n \subset \mathbb{C}$ such that each $v_n^{(i)}$ is the ith element of the vector v_n. Since for any $\varepsilon > 0$, $\|v_n^{(i)} - v_m^{(i)}\|_p \le \|v_n - v_m\|_p \le \varepsilon$, $(v_n^{(i)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, then $\lim_{n\to\infty} v_n^{(i)} = v^{(i)} \in \mathbb{C}$.

Hence, the sequence (v_n) converges elementwise to

$$\lim_{n\to\infty} v_n \;=\; \begin{bmatrix} v^{(0)} & v^{(1)} & \ldots & v^{(N-1)} \end{bmatrix}^\top \;=\; v \in \mathbb{C}^N.$$

This implies that $\mathbb{C}^N$ is a complete vector space.

2.24. *Cauchy sequences*
Let (x_n) be a convergent sequence in a normed vector space V and denote its limit by x. Then

$$\begin{aligned}\lim_{n,m\to\infty}\|x_n - x_m\| \;&=\; \lim_{n,m\to\infty}\|x_n - x + x - x_m\| \\ &\le\; \lim_{n\to\infty}\|x - x_n\| + \lim_{m\to\infty}\|x - x_m\| \;=\; 0.\end{aligned}$$

Every convergent sequence is thus a Cauchy sequence.

2.25. *Norms of operators*

(i) The eigenvalues of the matrix A are $\lambda_0 = 4$ and $\lambda_1 = -2$. The corresponding orthonormal eigenvectors are $v_0 = (1/\sqrt{2})\begin{bmatrix}-1 & 1\end{bmatrix}^\top$ and $v_1 = (1/\sqrt{2})\begin{bmatrix}1 & 1\end{bmatrix}^\top$. Any vector x can be decomposed as

$$x \;=\; \alpha_0\, v_0 + \alpha_1\, v_1. \tag{S2.25-1}$$

Because of the orthogonality of Av_0 and Av_1,

$$\|Ax\|^2 \;=\; \|A\alpha_0 v_0\|^2 + \|A\alpha_1 v_1\|^2 \;=\; \alpha_0^2\lambda_0^2\|v_0\|^2 + \alpha_1^2\lambda_1^2\|v_1\|^2 \;=\; \alpha_0^2\lambda_0^2 + \alpha_1^2\lambda_1^2.$$

From (S2.25-1), and for $\|x\| = 1$, we have $\alpha_0^2 + \alpha_1^2 = 1$. Therefore, we can write the norm of A as

$$\|A\|^2 \;=\; \sup_{\|x\|=1}\|Ax\|^2 \;=\; \sup_{\|x\|=1} \lambda_0^2\alpha_0^2 + \lambda_1^2(1-\alpha_0^2).$$

For $\|x\| = 1$, $0 \le \alpha_0^2 \le 1$. The above is maximized for α_0^2 being either 0 or 1; the choice is made by choosing the option with the maximum eigenvalue of A. Thus, $\|A\| = 4$.

The eigenvalues of the matrix A^{-1} are λ_0^{-1} and λ_1^{-1}. Therefore, the same analysis can be applied and the norm of the matrix A^{-1} is the absolute value of the maximum eigenvalue of the matrix A^{-1}. Thus, $\|A^{-1}\| = \frac{1}{2}$.

(ii) For any $x \in \ell^2(\mathbb{Z})$,

$$\|Ax\|^2 \;=\; \sum_n |(Ax)_n|^2 \;=\; \sum_n |e^{j\Theta_n}x_n|^2 \;\overset{(a)}{=}\; \sum_n |x_n|^2 \;=\; \|x\|^2 \;=\; 1,$$

where (a) follows from $|e^{j\Theta_n}| = 1$, for all n. Thus, $\|A\| = 1$.

(iii) Let $x \in \ell^2(\mathbb{Z})$ and $y = Ax$. We can write

$$\begin{bmatrix} y_{2n} \\ y_{2n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{2n} \\ x_{2n+1} \end{bmatrix}.$$

We see that

$$|y_{2n}|^2 + |y_{2n+1}|^2 = |x_{2n} + x_{2n+1}|^2 + |x_{2n} - x_{2n+1}|^2 = 2(|x_{2n}|^2 + |x_{2n+1}|^2).$$

From this it is clear that $\|y\|^2 = 2\|x\|^2$. Therefore, $\|A\| = \sqrt{2}$.

2.26. *Relation between operator norm and eigenvalues*

$$\begin{aligned} \|A\| &\overset{(a)}{=} \max_{\|x\|=1} \|Ax\| \overset{(b)}{=} \max_{\|x\|=1} (x^* A^* A x)^{1/2} \\ &\overset{(c)}{=} \max_{\|x\|=1} (x^* U \Lambda U^* x)^{1/2} = \max_{\|x\|=1} ((U^*x)^* \Lambda (U^* x))^{1/2} \\ &\overset{(d)}{=} \max_{\|y\|=1} (y^* \Lambda y)^{1/2} \overset{(e)}{=} \max_{\|y\|=1} \left(\textstyle\sum_{i=0}^{N-1} \lambda_i |y_i|^2 \right)^{1/2}, \end{aligned}$$

where (a) follows from the definition of operator norm, (2.45); (b) from the definition of 2 norm, (2.26a); (c) from the unitary diagonalization $A^*A = U\Lambda U^*$, (2.241a); (d) from using $y = U^*x$ and $\|y\| = \|x\|$ since U is unitary; and (e) from Λ being a diagonal matrix. In the diagonalization in step (c), we may assume without loss of generality that the diagonal matrix Λ has nonincreasing diagonal entries. The solution to the final maximization problem is achieved by $y = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^\top$, yielding $\|A\| = \sqrt{\lambda_{\max}(A^*A)}$ as desired.

2.27. *Adjoint operators*

(iv) For any x and y in H_1,

$$\langle AA^*x, y \rangle = \langle A(A^*x), y \rangle \overset{(a)}{=} \langle A^*x, A^*y \rangle \overset{(b)}{=} \langle x, AA^*y \rangle,$$

where (a) follows from A^* being the adjoint of A; and (b) from A being the adjoint of A^*. Hence, the adjoint of AA^* is AA^*. A similar computation shows that A^*A is self-adjoint.

(vi) For any x and y in H_1,

$$\langle x, y \rangle \overset{(a)}{=} \langle AA^{-1}x, y \rangle = \langle A(A^{-1}x), y \rangle \overset{(b)}{=} \langle A^{-1}x, A^*y \rangle \overset{(c)}{=} \langle x, ((A^{-1})^* A^* y \rangle,$$

where (a) follows from the definition of inverse; and (b) and (c) from the definition of the adjoint. Since this holds for every x and y in H_1, we have shown that $(A^{-1})^*$ is a left inverse of A^*. A similar computation shows that $(A^{-1})^*$ is a right inverse of A^*. Thus invertibility of A implies the invertibility of A^* and $(A^*)^{-1} = (A^{-1})^*$.

(vii) For any x in H_0 and y in H_1,

$$\langle (A+B)x, y \rangle \overset{(a)}{=} \langle Ax, y \rangle + \langle Bx, y \rangle \overset{(b)}{=} \langle x, A^*y \rangle + \langle x, B^*y \rangle \overset{(c)}{=} \langle x, (A^* + B^*)y \rangle,$$

where (a) follows from additivity; (b) from A^* and B^* being the adjoints of A and B; and (c) from additivity. Since the adjoint is unique, we have $(A+B)^* = A^* + B^*$.

(viii) For any x in H_0 and y in H_2,

$$\langle BAx, y \rangle \overset{(a)}{=} \langle Ax, B^*y \rangle \overset{(b)}{=} \langle x, A^*B^*y \rangle,$$

where (a) follows from B^* being the adjoint of B; and (b) from A^* being the adjoint of A. Since the adjoint is unique, we have $(BA)^* = A^*B^*$.

2.28. *Eigenvalues of definite operators*

Let (λ, v) be an eigenpair of a self-adjoint operator A, that is, $Av = \lambda v$, $v \neq 0$. Thus:

$$v^*Av = v^*\lambda v = \lambda \underbrace{\|v\|_2^2}_{>0}.$$

(i) A positive semidefinite ($x^*Ax \geq 0$ for all x) implies that $\lambda \geq 0$.
A positive definite ($x^*Ax > 0$ for all $x \neq 0$) implies that $\lambda > 0$.

(ii) From (i), the statement holds by contraposition.

2.29. *Operator expansion*

(i) $I - A$ is not invertible if and only if there exists $x \neq 0$ such that

$$(I - A)\,x \;=\; 0.$$

This amounts to 1 being an eigenvalue of A, impossible by definition ($\|A\| < 1$).

(ii)

$$(I - A)\sum_{k=0}^{\infty} A^k y \;=\; \sum_{k=0}^{\infty} A^k y - \sum_{k=1}^{\infty} A^k y \;=\; A^0 y + \sum_{k=1}^{\infty} A^k y - \sum_{k=1}^{\infty} A^k y \;=\; y.$$

Since $(I - A)$ is invertible, multiplying both sides of the equation by $(I - A)^{-1}$ proves the identity.

(iii) For $\|y\| = 1$, the task is to bound the error,

$$\begin{aligned}
\varepsilon_K \;&=\; \left\|(I - A)^{-1} y - \sum_{k=0}^{K-1} A^k y\right\| \;=\; \left\|\sum_{k=0}^{\infty} A^k - \sum_{k=0}^{K-1} A^k\right\| \;=\; \left\|\sum_{k=K}^{\infty} A^k\right\| \\
&\leq\; \sum_{k=K}^{\infty} |\lambda_{\max}|^k \;=\; \frac{|\lambda_{\max}|^K}{1 - |\lambda_{\max}|},
\end{aligned}$$

where $\lambda_{\max}$ is the maximum eigenvalue of A. The error decays as $\lambda_{\max}^K$ (because $\|A\| < 1$ and thus $|\lambda_{\max}| < 1$), where K is the number of terms in the approximation.

2.30. *Projection via domain restriction*

(i) To show that $1_{\mathcal{I}}$ is an orthogonal projection operator, we can either directly show that the error is orthogonal to the projection operator, or we can show that it is idempotent and self-adjoint, and thus an orthogonal projection operator.

The first approach yields

$$\begin{aligned}
\langle x - 1_{\mathcal{I}} x,\, 1_{\mathcal{I}} x\rangle \;&=\; \int_{-\infty}^{\infty} (x(t) - (1_{\mathcal{I}} x)(t))\,(1_{\mathcal{I}} x)(t)\,dt \\
&=\; \int_{-\infty}^{\infty} (x(t) - x(t)1_{\mathcal{I}}(t))\,x(t)1_{\mathcal{I}}(t)\,dt \\
&=\; \int_{-\infty}^{\infty} (x^2(t)1_{\mathcal{I}}(t) - x^2(t)\,\underbrace{1_{\mathcal{I}}(t)1_{\mathcal{I}}(t)}_{=1_{\mathcal{I}}(t)})\,dt \;=\; 0.
\end{aligned}$$

Thus, $1_{\mathcal{I}}$ is an orthogonal projection operator by Theorem 2.26.

The second approach yields,

$$\begin{aligned}
(1_{\mathcal{I}} 1_{\mathcal{I}} x)(t) \;&=\; 1_{\mathcal{I}}(t)1_{\mathcal{I}}(t)x(t) \\
&=\; 1_{\mathcal{I}}(t)x(t) \;=\; (1_{\mathcal{I}} x)(t),
\end{aligned}$$

so $1_{\mathcal{I}}$ is an idempotent operator. Furthermore,

$$\begin{aligned}
\langle 1_{\mathcal{I}} x,\, y\rangle \;&=\; \int_{-\infty}^{\infty} (1_{\mathcal{I}} x)(t)\,y(t)\,dt \\
&=\; \int_{-\infty}^{\infty} x(t)1_{\mathcal{I}}(t)\,y(t)\,dt \\
&=\; \int_{-\infty}^{\infty} x(t)\,(1_{\mathcal{I}} y)(t)\,dt \;=\; \langle x,\, 1_{\mathcal{I}} y\rangle,
\end{aligned}$$

so $1_{\mathcal{I}}$ is also self-adjoint. An idempotent, self-adjoint operator is an orthogonal projection operator.

(ii) Let $y_1 \in \mathcal{R}(\mathcal{I}_1)$, $y_2 \in \mathcal{R}(\mathcal{I}_2)$. In other words, there exist x_1, x_2 such that $y_1 = 1_{\mathcal{I}_1} x_1$ and $y_2 = 1_{\mathcal{I}_2} x_2$. Then,

$$\begin{aligned} \langle y_1, y_2 \rangle &= \int_{-\infty}^{\infty} (1_{\mathcal{I}_1} x_1)(t)\,(1_{\mathcal{I}_2} x_2)(t)\,dt \\ &= \int_{-\infty}^{\infty} x_1(t) x_2(t) 1_{\mathcal{I}_1}(t) 1_{\mathcal{I}_2}(t)\,dt \overset{(a)}{=} 0, \end{aligned}$$

where (a) follows from $1_{\mathcal{I}_1 \cap \mathcal{I}_2}(t) = 1_{\emptyset}(t)$. Since this holds for all $y_1 \in \mathcal{R}(\mathcal{I}_1)$, $y_2 \in \mathcal{R}(\mathcal{I}_2)$, the ranges of the associated operators, $\mathcal{R}(1_{\mathcal{I}_1})$ and $\mathcal{R}(1_{\mathcal{I}_2})$, are orthogonal.

(iii) $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathbb{R}$.

2.31. *Inverses, adjoints, and projections*
$A : H_0 \to H_1$ being a left inverse of $B : H_1 \to H_0$ means that AB is the identity on the Hilbert space H_1. Thus, for all $x \in H_0$,

$$BABAx = BIAx = BAx.$$

Hence BA is a bounded linear operator (by composition of bounded linear operators) that is idempotent, so it is a projection operator from H_0 onto H_1. Moreover if $B = A^*$, then

$$(BA)^* = A^*A = BA.$$

This projection operator is also self-adjoint, so it is an orthogonal projection operator.

2.32. *Projection operators*
From Theorem 2.29, we know that for $B : \mathbb{R}^2 \to \mathbb{R}^3$ and $A : \mathbb{R}^3 \to \mathbb{R}^2$ that is a left inverse of B, BA is a projection operator. Moreover, if $(BA)^* = BA$, then BA is an orthogonal projection operator. We use this to solve the exercise.

To find all left inverses of B, we write

$$AB = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = I.$$

From this,

$$a_{0,0} + a_{0,1} = 1, \quad a_{0,1} + a_{0,2} = 0, \quad a_{1,0} + a_{1,1} = 0, \quad a_{1,1} + a_{1,2} = 1.$$

Calling $a_{0,1} = \alpha$ and $a_{1,1} = \beta$, we get

$$A = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ -\beta & \beta & 1-\beta \end{bmatrix}.$$

Thus, projection operators BA are

$$BA = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ 1-(\alpha+\beta) & \alpha+\beta & 1-(\alpha+\beta) \\ -\beta & \beta & 1-\beta \end{bmatrix}.$$

We find orthogonal projection operators as those for which $(BA)^* = BA$, leading to $\beta = \alpha = \frac{1}{3}$,

$$A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad \text{and} \qquad BA = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

2.33. *Riesz bases*

(i) The standard basis $\{e_k\}_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ is a basis. To find whether this basis is a Riesz basis, we need to try to bound $\sum_{k\in\mathbb{Z}} |\langle x, e_k\rangle|^2$. Since

$$\sum_{k\in\mathbb{Z}} |\langle x, e_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |x_k|^2 = \|x\|^2,$$

it is a Riesz basis with optimal stability constants $\lambda_{\min} = \lambda_{\max} = 1$.

(ii) Scaling each basis vector by a finite nonzero scalar does not change the basis property because the basis vectors are still linearly independent, and, for any $x \in \ell^2(\mathbb{Z})$,

$$x \;=\; \sum_{k\in\mathbb{Z}} x_k e_k \;=\; \sum_{k\in\mathbb{Z}} x_k 2^{-k} \varphi_k \;=\; \sum_{k\in\mathbb{Z}} \alpha_k \varphi_k,$$

and thus, there is an expansion using $\{\varphi_k\}_{k\in\mathbb{Z}}$, with $\alpha_k = 2^{-k} x_k$.

To find whether this basis is a Riesz basis, we need to try to bound $\sum_{k\in\mathbb{Z}} |\alpha_k|^2$. Let $x = e_n$ for some $n \in \mathbb{Z}$. Then, the expansion of x with respect to $\{\varphi_k\}_{k\in\mathbb{Z}}$ has one term only: $x = \alpha_n \varphi_n$ with $\alpha_n = 2^{-n}$. Thus, $\sum_{k\in\mathbb{Z}} |\alpha_k|^2 = 2^{-2n}$. Since n is an arbitrary integer, taking $n \to \infty$ shows that there is no positive $\lambda_{\min}$ such that (2.89) holds, and taking $n \to -\infty$ shows that there is no finite $\lambda_{\max}$ such that (2.89) holds.

(iii) Since $0 < |\cos k| \le 1$, for all $k \in \mathbb{Z}$, the argument used in (ii) holds, and $\{\psi_k\}_{k\in\mathbb{Z}}$ is a basis for $\ell^2(\mathbb{Z})$, with

$$x \;=\; \sum_{k\in\mathbb{Z}} x_k e_k \;=\; \sum_{k\in\mathbb{Z}} x_k \frac{1}{\cos k} \psi_k \;=\; \sum_{k\in\mathbb{Z}} \beta_k \psi_k,$$

that is, there is an expansion using $\{\psi_k\}_{k\in\mathbb{Z}}$, with $\beta_k = (1/\cos k)\, x_k$.

To find whether this basis is a Riesz basis, we need to try to bound $\sum_{k\in\mathbb{Z}} |\beta_k|^2$. Since $0 < \cos^2 k \le 1$,

$$\sum_{k\in\mathbb{Z}} |\beta_k|^2 \;=\; \sum_{k\in\mathbb{Z}} \left| \frac{1}{\cos k} x_k \right|^2 \;=\; \sum_{k\in\mathbb{Z}} \frac{1}{\cos^2 k} |x_k|^2 \;\ge\; \sum_{k\in\mathbb{Z}} |x_k|^2 \;=\; \|x\|^2,$$

so the optimal lower stability constant is $\lambda_{\min} = 1$. However, no finite $\lambda_{\max}$ satisfies (2.89) because $\cos k$ can be arbitrarily close to zero.

2.34. *Basis that is not a Riesz basis*

The basis vectors $\{\varphi_k\}_{k\in\mathbb{N}}$ are still linearly independent, and, for any $x \in \ell^2(\mathbb{N})$,

$$\begin{aligned} x \;&=\; \sum_{k\in\mathbb{N}} x_k e_k \;\overset{(a)}{=}\; x_0\, \varphi_0 + \sum_{k=1}^{\infty} x_k \sqrt{k+1}\, (\varphi_k - \varphi_{k-1}) \\ &=\; \sum_{k=0}^{\infty} x_k \sqrt{k+1}\, \varphi_k - \sum_{k=1}^{\infty} x_k \sqrt{k+1}\, \varphi_{k-1} \\ &=\; \sum_{k\in\mathbb{N}} (\sqrt{k+1}\, x_k - \sqrt{k+2}\, x_{k+1})\, \varphi_k \;=\; \sum_{k\in\mathbb{Z}} \alpha_k \varphi_k, \end{aligned}$$

where (a) follows from

$$\varphi_k \;=\; \sum_{i=0}^{k} (i+1)^{-1/2} e_i \;=\; \varphi_{k-1} + (k+1)^{-1/2} e_k.$$

Thus, there is an expansion using $\{\varphi_k\}_{k\in\mathbb{Z}}$, with $\alpha_k = \sqrt{k+1}\, x_k - \sqrt{k+2}\, x_{k+1}$.

To find whether this basis is a Riesz basis, we need to try to bound $\sum_{k\in\mathbb{N}} |\alpha_k|^2$. Let $x = e_n$ for some $n \in \mathbb{N}$. Then the expansion of x with respect to $\{\varphi_k\}_{k\in\mathbb{N}}$ has two terms only: $x = \alpha_{n-1}\varphi_{n-1} + \alpha_n \varphi_n$ with $\alpha_{n-1} = -\sqrt{n+1}$ and $\alpha_n = \sqrt{n+1}$. Thus, $\sum_{k\in\mathbb{N}} |\alpha_k|^2 = 2(n+1)$. Since n is an arbitrary integer, taking $n \to \infty$ shows that there is no finite $\lambda_{\max}$ such that (2.89) holds.

2.35. *p norms in different bases*

(i) Under the 2 norm, $\mathbb{R}^2$ is a Hilbert space. Thus, $\|x\|_2 = \|\alpha\|_2$ is a case of the Parseval equality (2.96).

(ii) Let the basis be any orthonormal basis other than the standard basis and let $x = \varphi_0$. Then $(\alpha_0,\, \alpha_1) = (1,\, 0)$, so $\|\alpha\|_p = 1$ for any p. On the other hand, since the basis is orthonormal and not the standard basis, $|x_0|^2 + |x_1|^2 = 1$ with both of $|x_0|$ and $|x_1|$ in $(0, 1)$. For $p \in [1, 2)$,

$$|x_0|^p \;>\; |x_0|^2 \qquad \text{and} \qquad |x_1|^p \;>\; |x_1|^2,$$

so

$$\|x\|_p \;>\; \|x\|_2 \;=\; 1 \;=\; \|\alpha\|_p.$$

Also, for $p \in (2, \infty]$,

$$|x_0|^p \;<\; |x_0|^2 \qquad \text{and} \qquad |x_1|^p \;<\; |x_1|^2,$$

so

$$\|x\|_p \;<\; \|x\|_2 \;=\; 1 \;=\; \|\alpha\|_p.$$

(iii) Let $\{\varphi_0, \varphi_1\}$ and $\{\widetilde{\varphi}_0, \widetilde{\varphi}_1\}$ be a biorthogonal pair of bases for $\mathbb{R}^2$, where $\{\varphi_0, \varphi_1\}$ is not an orthonormal basis. Let $x = \widetilde{\varphi}_0$ and $y = \widetilde{\varphi}_1$. These have expansion coefficient vectors with respect to $\{\varphi_0, \varphi_1\}$ of $\alpha = (1, 0)$ and $\beta = (0, 1)$. Furthermore $x + y$ has expansion coefficient vector with respect to $\{\varphi_0, \varphi_1\}$ of $\alpha + \beta = (1, 1)$. Suppose invariance of the 2 norm holds, so

$$\|x\|_2^2 \;=\; \|\alpha\|_2^2 \;=\; 1, \tag{S2.35-1a}$$
$$\|y\|_2^2 \;=\; \|\beta\|_2^2 \;=\; 1, \tag{S2.35-1b}$$
$$\|x+y\|_2^2 \;=\; \|\alpha+\beta\|_2^2 \;=\; 2. \tag{S2.35-1c}$$

Since the vector space is real, expanding through linearity gives

$$\begin{aligned}\|x+y\|_2^2 \;&=\; \langle x+y,\, x+y\rangle \\ &=\; \langle x,\, x\rangle + \langle x,\, y\rangle + \langle y,\, x\rangle + \langle y,\, y\rangle \\ &=\; \|x\|_2^2 + \|y\|_2^2 + 2\langle x,\, y\rangle.\end{aligned} \tag{S2.35-2}$$

Substituting equations (S2.35-1) into (S2.35-2) gives $\langle x,\, y\rangle = 0$. This contradicts the bases not being orthogonal. Therefore, invariance of the 2 norm must not hold.

2.36. *Even and odd functions*

(i) We can express any $x(t) \in \mathcal{L}^2([-\pi, \pi])$ as

$$\begin{aligned}x(t) \;&=\; x(t) + \frac{1}{2}x(-t) - \frac{1}{2}x(-t) \;=\; \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t)) \\ &=\; x_{\text{even}}(t) + x_{\text{odd}}(t).\end{aligned}$$

Because

$$x_{\text{even}}(t) \;=\; x_{\text{even}}(-t) \qquad \text{and} \qquad x_{\text{odd}}(t) \;=\; -x_{\text{odd}}(-t),$$

$x_{\text{even}}(t) \in S_{\text{even}}$ and $x_{\text{odd}}(t) \in S_{\text{odd}}$.

(ii) An orthonormal basis for S_{even} is

$$\left\{\frac{1}{\sqrt{2\pi}},\, \frac{1}{\sqrt{\pi}}\cos(kt)\right\}_{k=1}^{\infty}$$

while an orthonormal basis for S_{odd} is

$$\left\{\frac{1}{\sqrt{\pi}}\sin(kt)\right\}_{k=1}^{\infty}.$$

(iii) This follows directly from (i) and (ii).

2.37. *Least-squares approximation with an orthonormal basis*

We write the error as

$$x - \widehat{x} \;=\; \sum_{i=0}^{k-1}(\alpha_i - \beta_i)\varphi_i + \sum_{i=k}^{N-1}\alpha_i\varphi_i.$$

Its norm squared is

$$\begin{aligned}\|x - \widehat{x}\|^2 \;&=\; \langle x - \widehat{x},\, x - \widehat{x}\rangle \\ &=\; \Big\langle \sum_{i=0}^{k-1}(\alpha_i - \beta_i)\varphi_i + \sum_{i=k}^{N-1}\alpha_i\varphi_i,\, \sum_{\ell=0}^{k-1}(\alpha_\ell - \beta_\ell)\varphi_\ell + \sum_{\ell=k}^{N-1}\alpha_\ell\varphi_\ell\Big\rangle.\end{aligned}$$

In the above, the two cross products are both 0 since they involve disjoint subsets of orthonormal vectors. What is left is

$$\begin{aligned}
\|x-\widehat{x}\|^2 &= \left\langle \sum_{i=0}^{k-1}(\alpha_i-\beta_i)\varphi_i,\, \sum_{\ell=0}^{k-1}(\alpha_\ell-\beta_\ell)\varphi_\ell \right\rangle + \left\langle \sum_{i=k}^{N-1}\alpha_i\varphi_i,\, \sum_{\ell=k}^{N-1}\alpha_\ell\varphi_\ell \right\rangle \\
&\overset{(a)}{=} \sum_{i,\ell=0}^{k-1}(\alpha_i-\beta_i)(\alpha_\ell-\beta_\ell)\langle\varphi_i,\,\varphi_\ell\rangle + \sum_{i,\ell=k}^{N-1}\alpha_i\alpha_\ell\langle\varphi_i,\,\varphi_\ell\rangle \\
&\overset{(b)}{=} \sum_{i,\ell=0}^{k-1}(\alpha_i-\beta_i)(\alpha_\ell-\beta_\ell)\delta_{i-\ell} + \sum_{i,\ell=k}^{N-1}\alpha_i\alpha_\ell\delta_{i-\ell} \\
&= \sum_{i=0}^{k-1}|\alpha_i-\beta_i|^2 + \sum_{i=k}^{N-1}|\alpha_i|^2,
\end{aligned}$$

where (a) follows from the linearity of the inner product; and (b) from the orthonormality of the set $\{\varphi_1,\,\varphi_2,\,\ldots,\,\varphi_N\}$. The last expression is minimized by $\beta_i=\alpha_i$ for $i=0,\,1,\,\ldots,\,k$.

2.38. *Biorthogonal pair of bases of cosine functions*

(i) To show that Ψ and $\widetilde{\Psi}$ satisfy the biorthogonality condition (2.111), we use how these sets are related to Φ, whose orthonormality has already been established in Example 2.34.

Clearly $\langle\psi_0,\,\widetilde{\psi}_0\rangle = 1$. Next,

$$\langle\psi_0,\,\widetilde{\psi}_k\rangle \;=\; 0, \qquad k=1,\,2,\,\ldots,$$

since each $\widetilde{\psi}_k$ is a linear combination of $\{\varphi_m\}_{m=1}^{k}$ and each element of $\{\varphi_m\}_{m=1}^{k}$ is orthogonal to φ_0. Similarly,

$$\langle\psi_k,\,\widetilde{\psi}_0\rangle \;=\; 0, \qquad k=1,\,2,\,\ldots,$$

since each ψ_k is a linear combination of φ_k and φ_{k+1}, each of which is orthogonal to $\widetilde{\psi}_0=\varphi_0$. Only the key case of

$$\langle\psi_k,\,\widetilde{\psi}_\ell\rangle, \qquad k=1,\,2,\,\ldots, \quad \ell=1,\,2,\,\ldots$$

remains. For k and ℓ both positive integers, we have

$$\begin{aligned}
\langle\psi_k,\,\widetilde{\psi}_\ell\rangle &= \left\langle \varphi_k+\tfrac{1}{2}\varphi_{k+1},\,\widetilde{\psi}_\ell \right\rangle \;=\; \left\langle \varphi_k+\tfrac{1}{2}\varphi_{k+1},\, \sum_{m=1}^{\ell}(-\tfrac{1}{2})^{\ell-m}\varphi_m \right\rangle \\
&= \sum_{m=1}^{\ell}(-\tfrac{1}{2})^{\ell-m}\left\langle \varphi_k+\tfrac{1}{2}\varphi_{k+1},\,\varphi_m \right\rangle \\
&= \sum_{m=1}^{\ell}(-\tfrac{1}{2})^{\ell-m}\left(\langle\varphi_k,\,\varphi_m\rangle + \tfrac{1}{2}\langle\varphi_{k+1},\,\varphi_m\rangle\right).
\end{aligned}$$

When $\ell > k > 0$,

$$\begin{aligned}
\langle\psi_k,\,\widetilde{\psi}_\ell\rangle &= \sum_{m=1}^{\ell}(-\tfrac{1}{2})^{\ell-m}\left(\delta_{k-m}+\tfrac{1}{2}\delta_{k+1-m}\right) \\
&= (-\tfrac{1}{2})^{\ell-k} + \tfrac{1}{2}(-\tfrac{1}{2})^{\ell-(k+1)} \;=\; (-\tfrac{1}{2})^{\ell-k}\left[1+\tfrac{1}{2}(-\tfrac{1}{2})^{-1}\right] \;=\; 0;
\end{aligned}$$

when $\ell = k > 0$,

$$\langle\psi_k,\,\widetilde{\psi}_\ell\rangle \;=\; \langle\varphi_k,\,(-\tfrac{1}{2})^0\varphi_k\rangle \;=\; 1;$$

and when $\ell < k$, the inner product is zero. This shows that Ψ and $\widetilde{\Psi}$ satisfy the biorthogonality conditions (2.111).

(ii) Since the basis functions of Ψ and $\widetilde{\Psi}$ are linear combinations of basis functions of Φ, it follows from the definition of the closure of a span that $\overline{\text{span}}(\Psi) \subseteq \overline{\text{span}}(\Phi)$ and $\overline{\text{span}}(\widetilde{\Psi}) \subseteq \overline{\text{span}}(\Phi)$. Therefore, we need to show $\overline{\text{span}}(\Phi) \subseteq \overline{\text{span}}(\Psi)$ and $\overline{\text{span}}(\Phi) \subseteq \overline{\text{span}}(\widetilde{\Psi})$ to prove the equality. We have

$$\varphi_k \;=\; \sum_{\ell=k}^{\infty} \left(-\tfrac{1}{2}\right)^{\ell-k} \psi_\ell \;=\; \sum_{\ell=1}^{k} \widetilde{\psi}_k - \tfrac{1}{2}\widetilde{\psi}_{k-1},$$

for $k > 0$. Since $\varphi_0 = \psi_0 = \widetilde{\psi}_0$, we conclude that $\overline{\text{span}}(\Phi) = \overline{\text{span}}(\Psi) = \overline{\text{span}}(\widetilde{\Psi})$.

2.39. *Dual bases*

(i) According to Theorem 2.46, given Φ, its unique dual is

$$\widetilde{\Phi} \;=\; \Phi(\Phi^*\Phi)^{-1}.$$

Its dual is then

$$\begin{aligned}
\widetilde{\widetilde{\Phi}} \;&=\; \widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1} \;=\; \Phi(\Phi^*\Phi)^{-1}((\Phi(\Phi^*\Phi)^{-1})^*(\Phi(\Phi^*\Phi)^{-1}))^{-1} \\
&=\; \Phi(\Phi^*\Phi)^{-1}(((\Phi^*\Phi)^{-1})^*\Phi^*\Phi(\Phi^*\Phi)^{-1})^{-1} \\
&=\; \Phi(\Phi^*\Phi)^{-1}(((\Phi^*\Phi)^*)^{-1})^{-1} \;=\; \Phi(\Phi^*\Phi)^{-1}\Phi^*\Phi \;=\; \Phi.
\end{aligned}$$

(ii) We can express the statement that the dual of Φ is Φ as

$$\Phi(\Phi^*\Phi)^{-1} \;=\; \Phi.$$

Since Φ is a basis by assumption, we can multiply both sides by the inverse of Φ,

$$\Phi^*\Phi \;=\; I,$$

or, in other words, Φ is unitary (orthonormal basis).

(iii) We use (2.113b) to write (2.89) as follows:

$$\lambda_{\min}\, x^*x \;\leq\; (\widetilde{\Phi}^*\, x)^*(\widetilde{\Phi}^*\, x) \;=\; x^*\widetilde{\Phi}\widetilde{\Phi}^*x \;\leq\; \lambda_{\max}\, x^*x.$$

Similarly, for the dual basis, we want to bound $x^*\Phi\Phi^*x$. Since $\Phi\Phi^*$ is a positive definite matrix, according to (2.243), it can be bounded from below and above by its minimum and maximum eigenvalues. Because

$$\Phi\Phi^* \;=\; (\widetilde{\Phi}^*)^{-1}\widetilde{\Phi}^{-1} \;=\; (\widetilde{\Phi}\widetilde{\Phi}^*)^{-1},$$

and the eigenvalues of $\Phi\Phi^*$ and its inverse are inverses of each other,

$$\frac{1}{\lambda_{\max}}\, x^*x \;\leq\; x^*\Phi\Phi^*x \;\leq\; \frac{1}{\lambda_{\min}}\, x^*x.$$

2.40. *Oblique projection property*

$P_{\mathcal{I}}$ is clearly a linear operator on H with range contained in $S_{\mathcal{I}}$. The idempotency of $P_{\mathcal{I}}$ can be proven with a computation closely following the proof of Theorem 2.39: For any $x \in H$,

$$\begin{aligned}
P_{\mathcal{I}}(P_{\mathcal{I}}\, x) \;&\overset{(a)}{=}\; \Phi_{\mathcal{I}}\, \widetilde{\Phi}_{\mathcal{I}}^*\, (\Phi_{\mathcal{I}}\, \widetilde{\Phi}_I^*\, x) \;\overset{(b)}{=}\; \Phi_{\mathcal{I}}\, (\widetilde{\Phi}_{\mathcal{I}}^*\, \Phi_{\mathcal{I}})\, \widetilde{\Phi}_I^*\, x \\
&\overset{(c)}{=}\; \Phi_{\mathcal{I}}\, (I)\, \widetilde{\Phi}_I^*\, x \;=\; \Phi_{\mathcal{I}}\, \widetilde{\Phi}_I^*\, x \;\overset{(d)}{=}\; P_{\mathcal{I}}\, x,
\end{aligned}$$

where (a) follows from (2.134b); (b) from associativity; (c) from the analogue of (2.123) for sequences in $\ell^2(\mathcal{I})$; and (d) from (2.134b). This shows that $P_{\mathcal{I}}$ is a projection operator. Note that, unlike in Theorem 2.39, we do not expect $P_{\mathcal{I}}$ to be self-adjoint.

The desired orthogonality relation for the residual follows from (2.111), (2.114a), and (2.134a). Specifically, since (2.114a) and (2.134a) give

$$x - P_{\mathcal{I}}\, x \;=\; \sum_{k\in\mathcal{K}\setminus\mathcal{I}} \langle x,\, \widetilde{\varphi}_k\rangle \varphi_k$$

and (2.111) gives $\{\varphi_k\}_{k\in\mathcal{K}\setminus\mathcal{I}} \perp \{\widetilde{\varphi}_k\}_{k\in\mathcal{I}}$, we must have $x - P_{\mathcal{I}}\, x \perp \widetilde{S}_{\mathcal{I}}$.

2.41. *Orthogonal projection in coefficient space*

(i) Since Ψ is a basis for H, we have

$$\widehat{\alpha} \;=\; \widetilde{\Psi}^* \widehat{x}. \tag{S2.41-1}$$

On the other hand, $\widehat{x}$ is the orthogonal projection of x onto $\overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{I}})$, and thus, we can use (2.139) to write

$$\widehat{x} \;=\; \Phi(\Phi^*\Phi)^{-1}\Phi^* x. \tag{S2.41-2}$$

Combining (S2.41-1), (S2.41-2) and $x = \Psi\alpha$, we have

$$\widehat{\alpha} \;=\; \widetilde{\Psi}^*\widehat{x} \;=\; \widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^* x \;=\; \widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi\,\alpha \;=\; P\,\alpha.$$

(ii) To show that P is a projection operator, we check idempotency,

$$\begin{aligned}
P^2 \;&=\; \widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi\widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi \\
&\overset{(a)}{=}\; \widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi \\
&\overset{(b)}{=}\; \widetilde{\Psi}^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi \;=\; P,
\end{aligned}$$

where (a) follows because Ψ is a basis; and (b) from $\Phi^*\Phi(\Phi^*\Phi)^{-1} = I$.

(iii) If $\{\psi_k\}_{k\in\mathcal{K}}$ is an orthonormal basis, then $\widetilde{\Psi} = \Psi$, $P = \Psi^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi$, and thus

$$P^* \;=\; (\Psi^*\Phi(\Phi^*\Phi)^{-1}\Phi^*\Psi)^* \;=\; P,$$

implying that P is an orthogonal projection operator.

If P is an orthogonal projection operator, then from $P^* = P$ we must have $\widetilde{\Psi} = \Psi$, and thus, $\{\psi_k\}_{k\in\mathcal{K}}$ is an orthonormal basis.

2.42. *Successive approximation with nonorthogonal basis*

Normal equations (2.138a) state that

$$\Phi^*\widehat{x} \;=\; \Phi^* x.$$

We use induction to prove the statement:

(i) For $k = 1$,

$$\begin{aligned}
\Phi^{(1)} \;&=\; \begin{bmatrix}\varphi_0\end{bmatrix}, \quad v_0 \;=\; 0, \quad \psi_0 \;=\; \frac{\varphi_0}{\|\varphi_0\|}, \\
\widehat{x}^{(1)} \;&=\; \widehat{x}^{(0)} + \langle x,\,\varphi_0\rangle\frac{\varphi_0}{\|\varphi_0\|^2} \;=\; \langle x,\,\varphi_0\rangle\frac{\varphi_0}{\|\varphi_0\|^2},
\end{aligned}$$

and thus

$$\begin{aligned}
(\Phi^{(1)})^*\,\widehat{x}^{(1)} \;&=\; \langle\varphi_0,\,\widehat{x}^{(1)}\rangle \;=\; \langle\varphi_0,\,\langle x,\,\varphi_0\rangle\frac{\varphi_0}{\|\varphi_0\|^2}\rangle \\
&=\; \frac{1}{\|\varphi_0\|^2}\langle x,\,\varphi_0\rangle\langle\varphi_0,\,\varphi_0\rangle \;=\; \langle x,\,\varphi_0\rangle \;=\; \Phi^{(1)*}\,x.
\end{aligned}$$

(ii) For $k = n$, we assume that the normal equations are satisfied,

$$(\Phi^{(n)})^*\,\widehat{x}^{(n)} \;=\; (\Phi^{(n)})^*\,x,$$

with

$$\Phi^{(n)} \;=\; \begin{bmatrix}\varphi_0 & \varphi_1 & \ldots & \varphi_{n-1}\end{bmatrix}.$$

For $k = n+1$,

$$\widehat{x}^{(n+1)} \;=\; \widehat{x}_n + \langle x,\,\psi_n\rangle\psi_n,$$

and thus,

$$\begin{aligned}
(\Phi^{(n+1)})^*\,\widehat{x}^{(n+1)} \;&=\; \begin{bmatrix}(\Phi^{(n)})^* \\ \varphi_n^*\end{bmatrix}\left(\widehat{x}^{(n)} + \langle x,\,\psi_n\rangle\psi_n\right) \\
&=\; \begin{bmatrix}(\Phi^{(n)})^*\,\widehat{x}^{(n)} \;+\; \langle x,\,\psi_n\rangle\,(\Phi^{(n)})^*\,\psi_n \\ \langle\varphi_n,\,\widehat{x}^{(n)}\rangle \;+\; \langle x,\,\psi_n\rangle\langle\varphi_n,\,\psi_n\rangle\end{bmatrix}.
\end{aligned}$$

By construction, $(\Phi^{(n)})^*\psi_n = 0$, and thus, the first element of the vector is

$$(\Phi^{(n)})^*\,\widehat{x}^{(n)} \;=\; (\Phi^{(n)})^*\,x,$$

where the equality follows from the assumption.

For the second element of the vector, we decompose φ_n as

$$\varphi_n \;=\; v_n + \gamma_n,$$

where v_n is the orthogonal projection of φ_n onto $\operatorname{span}\{\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_{n-1}\}$ and γ_n is orthogonal to $\operatorname{span}\{\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_{n-1}\}$. Moreover, $\gamma_n \;=\; \|\varphi_n - v_n\|\,\psi_n$. Thus the second element of the vector is

$$\begin{aligned}
&\langle \varphi_n,\, \widehat{x}^{(n)}\rangle + \langle x,\, \psi_n\rangle\langle \varphi_n,\, \psi_n\rangle \\
&= \;\langle v_n + \gamma_n,\, \widehat{x}^{(n)}\rangle \;+\; \langle x,\, \psi_n\rangle\langle v_n + \gamma_n,\, \psi_n\rangle \\
&= \;\langle v_n,\, \widehat{x}^{(n)}\rangle + \langle \gamma_n,\, \widehat{x}^{(n)}\rangle + \langle x,\, \psi_n\rangle\langle v_n,\, \psi_n\rangle + \langle x,\, \psi_n\rangle\langle \gamma_n,\, \psi_n\rangle \\
&\overset{(a)}{=} \;\langle v_n,\, \widehat{x}^{(n)}\rangle + \langle \gamma_n,\, x\rangle \\
&\overset{(b)}{=} \;\langle v_n,\, \Phi^{(n)}(\Phi^{(n)})^*\widehat{x}^{(n)}\rangle + \langle \gamma_n,\, x\rangle \\
&\overset{(c)}{=} \;\langle v_n,\, \Phi^{(n)}(\Phi^{(n)})^*x\rangle + \langle \gamma_n,\, x\rangle \\
&\overset{(d)}{=} \;\langle v_n,\, x\rangle + \langle \gamma_n,\, x\rangle \\
&= \;\langle v_n + \gamma_n,\, x\rangle \;=\; \varphi_n x,
\end{aligned}$$

where (a) follows from $\widehat{x}^{(n)}, v_n \in \operatorname{span}\{\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_{n-1}\}$ and γ_n orthogonal to $\operatorname{span}\{\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_{n-1}\}$ implies that γ_n is orthogonal to $\widehat{x}^{(n)}$; (b) from $\widehat{x}^{(n)}$ belongs to $\operatorname{span}\{\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_{n-1}\}$ implies that $\widehat{x}^{(n)} = \Phi^{(n)}(\Phi^{(n)})^*\,\widehat{x}^{(n)}$; (c) from the assumption on $k = n$; and (d) from $v_n \in \operatorname{span}\{\Phi^{(n)}\}$ implies that $v_n^*\Phi^{(n)}(\Phi^{(n)})^*x \;=\; v_n^*x$.

Finally, we have

$$(\Phi^{(n+1)})^*\,\widehat{x}^{(n+1)} \;=\; \begin{bmatrix} (\Phi^{(n)})^* \\ \varphi_n^* \end{bmatrix} x \;=\; (\Phi^{(n+1)})^*\,x.$$

2.43. *Exploring the definition of a frame*

(i) We have

$$\varphi_j \;=\; \sum_{k\in\mathcal{J}\setminus\{j\}} \beta_k\varphi_k, \tag{S2.43-1}$$

for some $\beta \in \ell^2(\mathcal{J})$ with $\beta_j = 0$. Let x be any vector in H. Since $\{\varphi_k\}_{k\in\mathcal{J}}$ is a frame, there exists an expansion

$$x \;=\; \sum_{k\in\mathcal{J}} \alpha_k\varphi_k, \tag{S2.43-2}$$

for some $\alpha \in \ell^2(\mathcal{J})$. Then, for any $c \in \mathbb{R}$,

$$\begin{aligned}
x \;&\overset{(a)}{=}\; \sum_{k\in\mathcal{J}} \alpha_k\varphi_k \;=\; c\,\mathbf{0} + \sum_{k\in\mathcal{J}} \alpha_k\varphi_k \\
&\overset{(b)}{=}\; c\left(\varphi_j - \sum_{k\in\mathcal{J}\setminus\{j\}} \beta_k\varphi_k\right) + \sum_{k\in\mathcal{J}} \alpha_k\varphi_k \\
&\overset{(c)}{=}\; (c+\alpha_j)\varphi_j + \sum_{k\in\mathcal{J}\setminus\{j\}} (\alpha_k - c\beta_k)\varphi_k,
\end{aligned}$$

where (a) follows from (S2.43-2); (b) from using (S2.43-1) to substitute for $\mathbf{0}$; and (c) from grouping terms. Since the sum of squared magnitudes of the expansion coefficients is at least $|c+\alpha_j|^2$ and c is arbitrary, there is no finite $\lambda_{\max}$ such that (2.89) always holds.

(ii) In (i) we showed that not *all* expansion coefficient sequences with respect to a frame have a bounded ℓ^2 norm; we now show that an expansion coefficient sequence with a bounded ℓ^2 norm always exists.

Let Φ have optimal frame bounds $\lambda_{\min}$ and $\lambda_{\max}$, and let $\widetilde{\Phi}$ be the canonical dual frame defined in (2.160). Since Φ and $\widetilde{\Phi}$ are a dual pair of frames, an expansion with respect to Φ is given by analysis with $\widetilde{\Phi}$:

$$\alpha_k \;=\; \langle x,\, \widetilde{\varphi}_k\rangle, \qquad k \in \mathcal{J}.$$

We will show that this expansion satisfies (2.89).

First,

$$\sum_{k\in\mathcal{J}} |\alpha_k|^2 \;=\; \sum_{k\in\mathcal{J}} |\langle x,\, \widetilde{\varphi}_k\rangle|^2 \;\overset{(a)}{=}\; \sum_{k\in\mathcal{J}} \left|\langle x,\, (\Phi\Phi^*)^{-1}\varphi_k\rangle\right|^2 \;\overset{(b)}{=}\; \sum_{k\in\mathcal{J}} \left|\langle (\Phi\Phi^*)^{-1}x,\, \varphi_k\rangle\right|^2,$$

where (a) follows from (2.160b); and (b) from $(\Phi\Phi^*)^{-1}$ being self-adjoint. Now from the frame definition (2.142),

$$\lambda_{\min}\|(\Phi\Phi^*)^{-1}x\|^2 \;\le\; \sum_{k\in\mathcal{J}} |\alpha_k|^2 \;\le\; \lambda_{\max}\|(\Phi\Phi^*)^{-1}x\|^2, \tag{S2.43-3}$$

which is close to the desired form, but must be adjusted to have upper and lower bounds in terms of $\|x\|^2$.

From (2.147),

$$\lambda_{\max}^{-1}\, I \;\le\; (\Phi\Phi^*)^{-1} \;\le\; \lambda_{\min}^{-1}\, I.$$

Thus,

$$\lambda_{\max}^{-2}\, \|x\|^2 \;\le\; \|(\Phi\Phi^*)^{-1}x\|^2 \;\le\; \lambda_{\min}^{-2}\, \|x\|^2.$$

Combining this with (S2.43-3) gives

$$\lambda_{\min}\lambda_{\max}^{-2}\|x\|^2 \;\le\; \sum_{k\in\mathcal{J}} |\alpha_k|^2 \;\le\; \lambda_{\max}\lambda_{\min}^{-2}\|x\|^2,$$

which is of the desired form.

2.44. *Frame of cosine functions*

Since the frame elements are $\{\varphi_k\}_{k\in\mathbb{Z}} \cup \{\varphi_k^+\}_{k\in\mathbb{Z}}$, we seek the largest $\lambda_{\min}$ and smallest $\lambda_{\max}$ such that

$$\lambda_{\min}\, \|x\|^2 \;\le\; \sum_{k\in\mathbb{N}} \Big(|\langle x,\, \varphi_k\rangle|^2 + |\langle x,\, \varphi_k^+\rangle|^2\Big) \;\le\; \lambda_{\max}\, \|x\|^2,$$

for every x in $\overline{\operatorname{span}}(\Phi \cup \Phi^+)$. Since Φ is an orthonormal basis,

$$\sum_{k\in\mathbb{N}} |\langle x,\, \varphi_k\rangle|^2 \;=\; \|x\|^2, \qquad \text{for every } x \text{ in } \overline{\operatorname{span}}(\Phi \cup \Phi^+). \tag{S2.44-1}$$

Thus what remains is to find the greatest lower bound and least upper bound for

$$\frac{1}{\|x\|^2} \sum_{k\in\mathbb{N}} |\langle x,\, \varphi_k^+\rangle|^2, \qquad \text{for every } x \text{ in } \overline{\operatorname{span}}(\Phi \cup \Phi^+).$$

Since Φ is an orthonormal basis for $\overline{\operatorname{span}}(\Phi \cup \Phi^+)$, any $x \in \overline{\operatorname{span}}(\Phi \cup \Phi^+)$ can be written as

$$x \;=\; \sum_{m\in\mathbb{N}} \alpha_m \varphi_m, \tag{S2.44-2}$$

where

$$\sum_{m\in\mathbb{N}} |\alpha_m|^2 \;=\; \|x\|^2. \tag{S2.44-3}$$

The quantity to be minimized and maximized is

$$\sum_{k\in\mathbb{N}} |\langle x, \varphi_k^+\rangle|^2 \overset{(a)}{=} \sum_{k\in\mathbb{N}} \left|\left\langle \sum_{m\in\mathbb{N}} \alpha_m\varphi_m, \varphi_k^+\right\rangle\right|^2 \overset{(b)}{=} \sum_{k\in\mathbb{N}} \left|\sum_{m\in\mathbb{N}} \alpha_m \langle\varphi_m, \varphi_k^+\rangle\right|^2, \tag{S2.44-4}$$

where (a) follows from (S2.44-2); and (b) from the linearity in the first argument of the inner product. So we compute $\langle\varphi_m, \varphi_k^+\rangle$ for all $(m,k)\in\mathbb{N}^2$:

$$\langle\varphi_m, \varphi_0^+\rangle \overset{(a)}{=} \langle\varphi_m, \varphi_1\rangle \overset{(b)}{=} \delta_{m-1}, \tag{S2.44-5a}$$

$$\langle\varphi_m, \varphi_1^+\rangle \overset{(c)}{=} \langle\varphi_m, \varphi_0 + \tfrac{1}{\sqrt{2}}\varphi_2\rangle \overset{(b)}{=} \delta_m + \tfrac{1}{\sqrt{2}}\delta_{m-2}, \tag{S2.44-5b}$$

$$\begin{aligned}\langle\varphi_m, \varphi_k^+\rangle &\overset{(c)}{=} \langle\varphi_m, \tfrac{1}{\sqrt{2}}\varphi_{k-1} + \tfrac{1}{\sqrt{2}}\varphi_{k+1}\rangle \\ &\overset{(b)}{=} \tfrac{1}{\sqrt{2}}\delta_{m-k+1} + \tfrac{1}{\sqrt{2}}\delta_{m-k-1}, \qquad \text{for } k = 2, 3, \ldots,\end{aligned} \tag{S2.44-5c}$$

where (a) follows from (2.144a); (b) from orthonormality of Φ; and (c) from (2.144b). Thus, substituting (S2.44-5) into (S2.44-4) yields

$$\begin{aligned}\sum_{k\in\mathbb{N}} |\langle x, \varphi_k^+\rangle|^2 &= |\alpha_1|^2 + \left|\alpha_0 + \tfrac{1}{\sqrt{2}}\alpha_2\right|^2 + \sum_{k=2}^{\infty}\left|\tfrac{1}{\sqrt{2}}\alpha_{k-1} + \tfrac{1}{\sqrt{2}}\alpha_{k+1}\right|^2. \\ &= |\alpha_0|^2 + \frac{3}{2}|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + \cdots \\ &\quad + \sqrt{2}\,|\alpha_0\alpha_2| + \sum_{k=2}^{\infty}|\alpha_{k-1}\alpha_{k+1}| \\ &\overset{(a)}{=} \|x\|^2 + \frac{1}{2}|\alpha_1|^2 + \sqrt{2}\,|\alpha_0\alpha_2| + \sum_{k=2}^{\infty}|\alpha_{k-1}\alpha_{k+1}|,\end{aligned} \tag{S2.44-6}$$

where (a) follows from (S2.44-3).

Finding the greatest lower bound is now simple. The expression in (S2.44-6) is bounded below by $\|x\|^2$, and this lower bound is achieved by $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \cdots = 0$. Combining this with (S2.44-1), we have shown $\lambda_{\min} = 2$.

Finding the least upper bound is more difficult. Introducing the Lagrange multiplier λ, define the Lagrange function

$$J(\alpha, \lambda) = \sum_{k\in\mathbb{N}} \alpha_k^2 + \frac{1}{2}\alpha_1^2 + \sqrt{2}\,\alpha_0\alpha_2 + \sum_{k=2}^{\infty}\alpha_{k-1}\alpha_{k+1} - \lambda\sum_{k\in\mathbb{N}}\alpha_k^2,$$

where we have removed the absolute values since a maximizing α will have nonnegative entries. We can optimize by finding stationary points of J. Thus, we compute the following partial derivatives:

$$\begin{aligned}\frac{\partial J}{\partial\alpha_0} &= \sqrt{2}\,\alpha_2 - 2(\lambda-1)\alpha_0, \\ \frac{\partial J}{\partial\alpha_1} &= \alpha_1 + \alpha_3 - 2(\lambda-1)\alpha_1, \\ \frac{\partial J}{\partial\alpha_2} &= \sqrt{2}\,\alpha_0 + \alpha_4 - 2(\lambda-1)\alpha_2, \\ \frac{\partial J}{\partial\alpha_\ell} &= \alpha_{\ell-2} + \alpha_{\ell+2} - 2(\lambda-1)\alpha_\ell, \qquad \ell = 3, 4, \ldots.\end{aligned}$$

By setting the partial derivatives to zero, we can parameterize all the candidate maximizing vectors by α_1/α_0, and λ. Making further computations analytically is difficult. Numerically, one can verify that $J(\alpha, \lambda)$ is maximized under constraint (S2.44-3) by setting $\alpha_1/\alpha_0 = 0$ The result is for $J(\alpha, \lambda)$ to approach 2 from below as $\lambda \to 2^-$. Combining this with (S2.44-1), we have $\lambda_{\max} = 3$.

2.45. *Dual frame*

(i) For Ψ to be a basis, it must be linearly independent. This is not the case because the components of each element of Ψ sum up to 0. Thus, all $x \in \mathbb{R}^4$ whose components do not sum up to 0 will not be in span(Ψ).

Φ is not linearly independent because the sum of the first and third elements in Φ is equal to the sum of the second and fourth.

(ii) Since F_1 contains the basis E, it is a frame. F_2 is a frame because it contains a linearly-independent set of four vectors. Call S_1 the synthesis operator associated with F_1 and S_2 the synthesis operator associated with F_2. We can compute the optimal frame bounds of F_1 and F_2 as the minimum and maximum eigenvalues of $S_1 S_1^*$ and $S_2 S_2^*$, respectively. They are $\lambda_{\min} = 1$ and $\lambda_{\max} = 5$ for F_1 and $\lambda_{\min} = \lambda_{\max} = 4$ for F_2. So F_2 is a tight frame.

(iii) We can use (2.160a) to find canonical dual frames to F_1 and F_2. The corresponding synthesis operators are

$$\begin{aligned}
\widetilde{S_1} &= (S_1 S_1^*)^{-1} S_1 = \frac{1}{15}\begin{bmatrix} 7 & 1 & 2 & 1 & 4 & 1 & -1 & -4 \\ 1 & 7 & 1 & 2 & -4 & 4 & 1 & -1 \\ 2 & 1 & 7 & 1 & -1 & -4 & 4 & 1 \\ 1 & 2 & 1 & 7 & 1 & -1 & -4 & 4 \end{bmatrix}, \\
\widetilde{S_2} &= (S_2 S_2^*)^{-1} S_2 = \frac{1}{4} S_2.
\end{aligned}$$

2.46. *Properties of dual pair of frames*

(i) Assume that

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_K \end{bmatrix}_{n \times K}, \qquad \operatorname{rank}(\Phi) = m, \; m < K, n,$$

and

$$\Phi \widetilde{\Phi}^* x = x \qquad x \in H.$$

Thus, for all $\alpha \in \mathbb{R}^m$

$$\begin{aligned}
\Phi \widetilde{\Phi}^* Q \alpha &\overset{(a)}{=} Q\alpha, \\
Q^* \Phi \widetilde{\Phi}^* Q &= I_m, \\
(Q^* \Phi \widetilde{\Phi}^* Q)^* &= I_m, \\
Q^* \widetilde{\Phi} \Phi^* Q &= I_m, \\
\widetilde{\Phi} \Phi^* Q &= Q, \\
\widetilde{\Phi} \Phi^* Q \alpha &= Q\alpha, \quad \alpha \in \mathbb{R}^m, \\
\widetilde{\Phi} \Phi^* x &= x, \qquad x \in H,
\end{aligned}$$

where in (a) $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix}_{n \times m}$ is a orthonormal basis for H.

(ii) We have that

$$\langle \widetilde{\alpha}, \beta \rangle = \langle \Phi^* x, \widetilde{\Phi}^* y \rangle \overset{(a)}{=} \langle x, \Phi \widetilde{\Phi}^* y \rangle \overset{(b)}{=} \langle x, y \rangle,$$

where (a) follows from $(\Phi^*)^* = \Phi$; and (b) from the fact that for all $y \in H$ we have $\Phi \widetilde{\Phi}^* y = y$.

(iii) Call $P = \Phi_{\mathcal{I}} \widetilde{\Phi}_{\mathcal{I}}^*$. To have a projection operator, we need $P^2 = P$. Choose $\widetilde{\Phi}_{\mathcal{I}}$ to be the canonical dual of $\Phi_{\mathcal{I}}$, as in (2.160a). Then

$$\begin{aligned}
P^2 &= \Phi_{\mathcal{I}}\widetilde{\Phi}_{\mathcal{I}}^*\Phi_{\mathcal{I}}\widetilde{\Phi}_{\mathcal{I}}^* \\
&= \Phi_{\mathcal{I}}\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\,\Phi_{\mathcal{I}}\right)^*\Phi_{\mathcal{I}}\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\,\Phi_{\mathcal{I}}\right)^* \\
&= \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\right)^*\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\right)^* \\
&\overset{(a)}{=} \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\,(\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\,(\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\right)^* \\
&\overset{(b)}{=} \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\left((\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*)^{-1}\right)^* \\
&\overset{(c)}{=} \Phi_{\mathcal{I}}\widetilde{\Phi}_{\mathcal{I}}^* \;=\; P,
\end{aligned}$$

where (a) follows because the inverse and Hermitian conjugation commute; (b) from $\left(\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\right)^{-1}\left(\Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^*\right) = I$; and (c) from the definition for the canonical dual, (2.160a). Thus, a sufficient condition is for $\widetilde{\Phi}_{\mathcal{I}}$ to be the canonical dual of $\Phi_{\mathcal{I}}$.

2.47. *Tight frame with nonequal-norm vectors*
The frame matrix corresponding to the given set of vectors is

$$\Phi \;=\; \begin{bmatrix} 0 & \cos\theta & -\cos\theta \\ \alpha & \sin\theta & \sin\theta \end{bmatrix}. \tag{S2.47-1}$$

For Φ to be a tight frame, the following must hold:

$$\Phi\,\Phi^\top \;=\; c\,I, \qquad c \neq 0. \tag{S2.47-2}$$

Substituting (S2.47-1) into (S2.47-2), we get

$$\Phi\,\Phi^\top \;=\; \begin{bmatrix} 2\cos^2\theta & 0 \\ 0 & \alpha^2 + 2\sin^2\theta \end{bmatrix} \;=\; \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}.$$

From this,

$$2\cos^2\theta \;=\; c, \tag{S2.47-3a}$$

$$\alpha^2 + 2\sin^2\theta \;=\; c. \tag{S2.47-3b}$$

Combining the two equations we get

$$2\cos^2\theta - 2\sin^2\theta \;=\; \alpha^2 \quad\Rightarrow\quad \cos(2\theta) \;=\; \frac{\alpha^2}{2} \quad\Rightarrow\quad |\alpha| \;=\; \sqrt{2\cos(2\theta)}, \tag{S2.47-4a}$$

where we have used the double-angle formula for cosine. Moreover,

$$0 \overset{(a)}{<} \cos(2\theta) \overset{(b)}{\leq} 1, \tag{S2.47-4b}$$

where in (a) cosine is positive because of (S2.47-4a) and cannot be 0 because $\alpha \neq 0$; and (b) follows from the definition of a cosine. From here, we get

$$0 \;<\; \cos(2\theta) \quad\Rightarrow\quad 2k\pi - \frac{\pi}{2} \;<\; 2\theta \;<\; 2k\pi + \frac{\pi}{2} \quad\Rightarrow\quad k\pi - \frac{\pi}{4} \;<\; \theta \;<\; k\pi + \frac{\pi}{4}, \tag{S2.47-4c}$$

for $k \in \mathbb{Z}$. In terms of α, substituting (S2.47-4a) into (S2.47-4b), we get

$$0 \;<\; |\alpha| \;\leq\; \sqrt{2}. \tag{S2.47-4d}$$

In summary, combining (S2.47-4), we get

$$\begin{aligned}
|\alpha| &\;=\; \sqrt{2\cos(2\theta)}, \quad \alpha \neq 0, \\
0 \;<\; |\alpha| &\;\leq\; \sqrt{2}, \\
k\pi - \frac{\pi}{4} \;<\; \theta &\;<\; k\pi + \frac{\pi}{4}, \qquad k \in \mathbb{Z}.
\end{aligned}$$

For a given θ in the allowed range, α is fixed, leading to a tight frame. If α is chosen first, it must be smaller than or equal to $\sqrt{2}$, because for $\alpha > \sqrt{2}$, the frame stops being tight.

2.48. *Tight frame of affine functions*
We use the Gram–Schmidt orthogonalization (see Table 2.1) on $\{\varphi_0(t),\, \varphi_1(t)\}$ twice; the first time we start from $\varphi_0(t)$, and the second from $\varphi_1(t)$. This way, we obtain two orthonormal bases that together form a tight frame.

(i) Call $\psi_0(t) = \varphi_0(t) = 1$.

(ii) Use the Gram–Schmidt orthogonalization to get $\psi_1(t)$ from $\varphi_1(t)$

$$\psi_1(t) \;=\; \frac{\varphi_1(t) - \langle \varphi_1(t),\, \psi_0(t)\rangle\, \psi_0(t)}{\|\varphi_1(t) - \langle \varphi_1(t),\, \psi_0(t)\rangle\, \psi_0(t)\|} \;=\; \frac{\sqrt{3}t - \langle \sqrt{3}t,\, 1\rangle}{\|\sqrt{3}t - \langle \sqrt{3}t,\, 1\rangle\|} \;=\; \sqrt{3}(2t-1).$$

(iii) Call $\psi_2(t) = \varphi_1(t) = \sqrt{3}t$.

(iv) Use the Gram–Schmidt orthogonalization to get $\psi_3(t)$ from $\varphi_0(t)$

$$\psi_3(t) \;=\; \frac{\varphi_0(t) - \langle \varphi_0(t),\, \psi_2(t)\rangle\, \psi_2(t)}{\|\varphi_0(t) - \langle \varphi_0(t),\, \psi_2(t)\rangle\, \psi_2(t)\|} \;=\; \frac{1 - \langle 1,\, \sqrt{3}t\rangle \sqrt{3}t}{\|1 - \langle 1,\, \sqrt{3}t\rangle \sqrt{3}t\|} \;=\; 2 - 3t.$$

The following is then a tight frame:

$$\Phi \;=\; \begin{bmatrix} 1 & \sqrt{3}(2t-1) & \sqrt{3}t & 2-3t \end{bmatrix}.$$

As this tight frame is a union of two orthonormal bases, we expect the optimal frame bounds to be 2. To confirm this, we construct the Gram matrix (2.121),

$$G \;=\; \Phi^*\Phi \;=\; \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \end{bmatrix}$$

The largest eigenvalue of the Gram matrix is in fact equal to 2. Thus, $\lambda \;=\; 2$.

Since this frame is a tight frame, its canonical dual frame will be by definition

$$\widetilde{\Phi} \;=\; \frac{1}{\lambda}\Phi \;=\; \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3}(2t-1) & \sqrt{3}t & 2-3t \end{bmatrix}.$$

2.49. *Complex multiplication*
The following computation

$$\alpha \;=\; a(d-c), \quad \beta \;=\; b(c+d), \quad \gamma \;=\; c(a+b),$$

requires 3 multiplications and 3 additions. With 2 more additions,

$$e \;=\; \gamma - \beta \quad \text{and} \quad f \;=\; \gamma + \alpha,$$

we achieve the desired result.

Note that, if one of the terms in the complex multiplication is a constant fixed ahead of time (as will be the case in the FFT computation in Chapter 3), 2 additions can be precomputed, leading to a complex multiplication with 3 real multiplications and additions, or 6 operations.

2.50. *Gaussian elimination*

(i) The system has a unique solution since y belongs to the range of A and the columns of A are linearly independent. We use (2.199) to get

$$\begin{bmatrix} 10 \\ 20 \\ -45 \end{bmatrix} \;=\; B^{(2)}B^{(1)}Ax \;=\; \begin{bmatrix} 1 & 0 & 3 \\ 0 & -5 & 10 \\ 0 & 0 & -15 \end{bmatrix} x,$$

with

$$B^{(1)} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad B^{(2)} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 5 \end{bmatrix}.$$

By back substitution, we solve $x_2 = 3$, $x_1 = 2$, and $x_0 = 1$.

(ii) This system has infinitely many solutions since y belongs to the range of A and the columns of A are not linearly independent ($\det(A) = 0$).

(iii) The system has no solution since y does not belong to the range of A (we cannot express $y = \sum_{k=0}^{2} \alpha_k a_k$, where a_k is the kth column of A).

2.51. *Kaczmarz's algorithm*

(i) The point x belongs to the hyperplane S_i if and only if $\langle r_i, x\rangle = y_i$, where r_i are the rows of A. Thus, x belongs to $\bigcap_{i=0}^{N-1} S_i$ if and only if $\langle r_i, x\rangle = y_i$, for all $i = 0, 1, \ldots, N-1$, or, $Ax = y$. The matrix A is square and of full rank (its rows are linearly independent), and thus $Ax = y$ has a unique solution. This means that the N hyperplanes intersect at a single point.

(ii) The key is to notice each step makes a correction that is orthogonal to past and future corrections,

$$\begin{aligned}
x^{(0)} &= x^{(-1)} + (y_0' - \langle x^{(-1)}, \gamma_0\rangle)\gamma_0 \\
x^{(1)} &= x^{(-1)} + (y_0' - \langle x^{(-1)}, \gamma_0\rangle)\gamma_0 + (y_1' - \langle x^{(-1)}, \gamma_1\rangle)\gamma_1 \\
&\ \vdots \\
x^{(N-1)} &= x^{(-1)} + \sum_{i=0}^{N-1} (y_i' - \langle x^{(-1)}, \gamma_i\rangle)\gamma_i.
\end{aligned}$$

After one sweep, $x^{(N-1)}$ verifies all the constraints. For $j = 1, 2, \ldots, N-1$,

$$\begin{aligned}
\langle x^{(N-1)}, r_j\rangle &= \langle x^{(-1)}, r_j\rangle + \sum_{i=0}^{N-1} (y_i' - \langle x^{(-1)}, \gamma_i\rangle)\langle \gamma_i, r_j\rangle, \\
&\overset{(a)}{=} \langle x^{(-1)}, r_j\rangle + \sum_{i=0}^{N-1} (y_i' - \langle x^{(-1)}, \gamma_i\rangle)\, \|r_j\| \delta_{i-j} \\
&= \langle x^{(-1)}, r_j\rangle + y_j - \langle x^{(-1)}, \|r_j\|\gamma_j\rangle \ = \ y_j,
\end{aligned}$$

where (a) follows from $\gamma_n = r_n/\|r_n\|$.

2.52. *Convergence of sequences*

The convergence of $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ to a and b, respectively, means that for any $\varepsilon > 0$ there exist numbers A_ε and B_ε such that

$$|a_k - a| < \varepsilon \quad \text{for every } k > A_\varepsilon \qquad \text{and} \qquad |b_\ell - b| < \varepsilon \quad \text{for every } \ell > B_\varepsilon. \tag{S2.52-1}$$

(i) Assume that $c \neq 0$; if not, the statement trivially holds. Let $\varepsilon > 0$. Then, from (S2.52-1), we know that there exists a number $A_{\varepsilon/|c|}$ such that $|a_k - a| < \varepsilon/|c|$ for every $k > A_{\varepsilon/|c|}$. Then,

$$|ca_k - ca| \ = \ |c|\,|a_k - a| \ < \ |c|\,\frac{\varepsilon}{|c|} \ = \ \varepsilon.$$

In other words,

$$\begin{aligned}
&\text{for any } \varepsilon > 0, \text{there exists a number } K_\varepsilon = A_{\varepsilon/|c|} \text{ such that} \\
&|ca_k - ca| < \varepsilon \quad \text{for every } k > K_\varepsilon.
\end{aligned}$$

(ii) Let $\varepsilon > 0$. Then, from (S2.52-1), we know that there exist numbers $A_{\varepsilon/2}$ and $B_{\varepsilon/2}$ such that $|a_k - a| < \varepsilon/2$ and $|b_k - b| < \varepsilon/2$, for every $k > \max(A_{\varepsilon/2}, B_{\varepsilon/2})$. Then,

$$|a_k + b_k - (a+b)| \ < \ |a_k - a| + |b_k - b| \ < \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon.$$

In other words,

$$\begin{aligned}
&\text{for any } \varepsilon > 0, \text{there exists a number } K_\varepsilon = \max(A_{\varepsilon/2}, B_{\varepsilon/2}) \text{ such that} \\
&|a_k + b_k - (a+b)| < \varepsilon \quad \text{for every } k > K_\varepsilon.
\end{aligned}$$

(iii) Let $\varepsilon > 0$ and

$$\delta \;=\; -\frac{|a|+|b|}{2} + \sqrt{\left(\frac{|a|+|b|}{2}\right)^2 + \varepsilon}.$$

Then, from (S2.52-1), we know that there exist numbers A_δ and B_δ such that $|a_k - a| < \delta$ and $|b_k - b| < \delta$, for every $k > \max(A_\delta, B_\delta)$. Then,

$$\begin{aligned}
|a_k b_k - ab| \;&=\; |(a_k - a)(b_k - b) + a(b_k - b) + b(a_k - a)| \\
&<\; |a_k - a|\,|b_k - b| + |a|\,|b_k - b| + |b|\,|a_k - a| \\
&<\; \delta(|a| + |b| + \delta) \;=\; \varepsilon.
\end{aligned}$$

In other words,

$$\begin{aligned}
&\text{for any } \varepsilon > 0, \text{there exists a number } K_\varepsilon = \max(A_\delta, B_\delta) \text{ such that} \\
&|a_k b_k - ab| < \varepsilon \quad \text{for every } k > K_\varepsilon.
\end{aligned}$$

(iv) Let $\varepsilon > 0$,

$$\gamma \;=\; \frac{|a|}{2}, \qquad \text{and} \qquad c \;=\; \frac{|a|+|b|}{|a|\gamma}.$$

Then, from (S2.52-1), we know that there exist numbers $A_{\varepsilon/c}$ and $B_{\varepsilon/c}$ such that $|a_k - a| < \varepsilon/c$ and $|b_k - b| < \varepsilon/c$, for every $k > \max(A_{\varepsilon/c}, B_{\varepsilon/c})$. Similarly, there exists a number A_γ such that $|a_k| > \gamma$ for all $k > A_\gamma$. Then,

$$\begin{aligned}
\left|\frac{b_k}{a_k} - \frac{b}{a}\right| \;&=\; \frac{|ab_k - ba_k|}{|aa_k|} \;=\; \frac{|ab_k - ba_k + ab - ab|}{|aa_k|} \\
&<\; \frac{|ab_k - ab| + |ba_k - ab|}{|aa_k|} \;=\; \frac{|a|\,|b_k - b| + |b|\,|a_k - a|}{|aa_k|} \\
&<\; \frac{|a|(\varepsilon/c) + |b|(\varepsilon/c)}{|a|\gamma} \;<\; c\,\frac{\varepsilon}{c} \;=\; \varepsilon.
\end{aligned}$$

In other words,

$$\begin{aligned}
&\text{for any } \varepsilon > 0, \text{there exists a number } K_\varepsilon = \max(A_{\varepsilon/c}, A_\gamma, B_{\varepsilon/c}) \text{ such that} \\
&\left|\frac{b_k}{a_k} - \frac{b}{a}\right| < \varepsilon \quad \text{for every } k > K_\varepsilon.
\end{aligned}$$

What we have done in (i)–(iv) can be generalized as follows: To study the convergence of $(f(a_k, b_k))_{k=0}^\infty$, where f is some continuous function, we find g such that $\lim_{\delta\to 0} g(\delta) = 0$, and for which

$$|f(a_k, b_k) - f(a, b)| \;<\; g(\delta).$$

Choose ℓ such that $|a_\ell - a| < \delta$, $|b_\ell - b| < \delta$, for every $\ell \geq \max(A_\delta, B_\delta)$. Then,

$$\begin{aligned}
&\text{for any } \varepsilon > 0, \text{there exists a number } K_\varepsilon = \max(A_{g^{-1}(\varepsilon)}, B_{g^{-1}(\varepsilon)}) \text{ such that} \\
&|f(a_k, b_k) - f(a, b)| < \varepsilon \quad \text{for every } k > K_\varepsilon,
\end{aligned}$$

proving the convergence of $(f(a_k, b_k))_{k=0}^\infty$ to $f(a, b)$.

2.53. *Convergence tests*

(i) Since for any $k \geq 2$,

$$c_k \;=\; \frac{k^2}{2k^4 - 3} \;=\; \frac{k^2}{k^4 + (k^4 - 3)} \;<\; \frac{k^2}{k^4} \;=\; \frac{1}{k^2} \;=\; a_k,$$

and the series $\sum_{k=1}^\infty a_k$ converges,

$$\sum_{k=1}^\infty a_k \;=\; \sum_{k=1}^\infty \frac{1}{k^2} \;=\; \frac{\pi^2}{12},$$

the series $\sum_{k=1}^\infty c_k$ converges as well.

(ii) Since for any $k \geq 1$,

$$c_k \;=\; \frac{\log k}{k} \;>\; \frac{1}{k} \;=\; a_k,$$

and the series $\sum_{k=1}^{\infty} a_k$ diverges,

$$\sum_{k=1}^{\infty} a_k \;=\; \sum_{k=1}^{\infty} \frac{1}{k},$$

the series $\sum_{k=1}^{\infty} c_k$ diverges as well.

(iii) Since

$$c_k \;=\; \frac{k^k}{k!} \;=\; \frac{k \cdot k \cdot \cdots \cdot k}{1 \cdot 2 \cdot \cdots \cdot k} \;\geq\; 1 \;=\; a_k,$$

and the series $\sum_{k=1}^{\infty} a_k$ diverges,

$$\sum_{k=1}^{\infty} a_k \;=\; \sum_{k=1}^{\infty} 1,$$

the series $\sum_{k=1}^{\infty} c_k$ diverges as well.

(iv) Using the ratio test for convergence,

$$\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| \;=\; \lim_{k\to\infty} \left| \frac{\frac{a^{k+1}}{(k+1)!}}{\frac{a^k}{k!}} \right| \;=\; \lim_{k\to\infty} \left| \frac{a}{k+1} \right| \;=\; 0,$$

and thus, the series $\sum_{k=1}^{\infty} c_k$ converges absolutely.

2.54. *Useful series*

(i) *Finite Geometric Series:* The proof is straightforward:

$$\frac{1-t^N}{1-t} \;=\; \frac{(1-t)(1+t+\cdots+t^{N-1})}{1-t} \;=\; \sum_{k=0}^{N-1} t^k.$$

(ii) *Geometric Series:* As $N \to \infty$, the series

$$S_N \;=\; \sum_{n=1}^{N} t^n \;=\; \frac{1-t^{N+1}}{1-t} - 1$$

converges for $|t| < 1$ to $t/(1-t)$.

(iii) *Power Series:* The power series $\sum_{k=1}^{\infty} a_k t^k$ converges if

$$\lim_{k\to\infty} \left| \frac{a_{k+1}\, t^{k+1}}{a_k\, t^k} \right| \;=\; \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k}\, t \right| \;<\; 1.$$

Hence, the series converges for

$$|t| \;<\; \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|.$$

(iv) *Taylor Series:* The kth derivative of $x(t) = 1/(1-t)$ is

$$x^{(k)}(t) \;=\; \frac{k!}{(1-t)^{k+1}}.$$

Hence, the Taylor series expansion of $x(t)$ is

$$x(t) \;=\; \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} \frac{k!}{(1-t_0)^{k+1}} + R_n \;=\; \sum_{k=0}^{n} \frac{(t-t_0)^k}{(1-t_0)^{k+1}} + \frac{(t-t_0)^{n+1}}{(1-\xi)^{n+2}}.$$

(v) *MacLaurin Series:* From the above Taylor series expansion of $x(t)$, we derive the MacLaurin series expansion:

$$x(t) \;=\; \sum_{k=0}^{n} t^k + \frac{t^{n+1}}{(1-\xi)^{n+2}}.$$

2.55. *Eigenvalues and eigenvectors*

(i) The characteristic polynomial of A is

$$\det(\lambda I - A) \;=\; \det \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} \;=\; (\lambda-1)^2 - 4 \;=\; (\lambda-3)(\lambda+1).$$

The eigenvalues of A are $\lambda_0 = -1$ and $\lambda_1 = 3$.

For $\lambda_0 = -1$, we solve $Ax = -x$,

$$\begin{aligned} x_0 + 2x_1 &= -x_0, \\ 2x_0 + x_1 &= -x_1, \end{aligned}$$

yielding $x_1 = -x_0$. We choose $x_0 = 1$, $x_1 = -1$ and normalize. The eigenvector associated with λ_0 is thus $v_0 = 1/\sqrt{2}\begin{bmatrix}1 & -1\end{bmatrix}^\top$.

Similarly, for $\lambda_1 = 3$, we solve $Ax = 3x$,

$$\begin{aligned} x_0 + 2x_1 &= 3x_0, \\ 2x_0 + x_1 &= 3x_1, \end{aligned}$$

yielding $x_0 = x_1$. We choose $x_0 = x_1 = 1$ and normalize. The eigenvector associated with λ_1 is thus $v_1 = 1/\sqrt{2}\begin{bmatrix}1 & 1\end{bmatrix}^\top$.

The characteristic polynomial of B is

$$\det(\lambda I - B) \;=\; (\lambda-\alpha)^2 - \beta^2 \;=\; (\lambda-(\alpha+\beta))(\lambda-(\alpha-\beta)).$$

The eigenvalues of B are $\lambda_0 = \alpha-\beta$ and $\lambda_1 = \alpha+\beta$.

For $\lambda_0 = \alpha-\beta$, we solve $Bx = (\alpha-\beta)x$, $\beta \neq 0$,

$$\begin{aligned} \alpha x_0 + \beta x_1 &= (\alpha-\beta)x_0, \\ \beta x_0 + \alpha x_1 &= (\alpha-\beta)x_1, \end{aligned}$$

yielding $x_1 = -x_0$. The eigenvector associated with λ_0 is thus $v_0 = 1/\sqrt{2}\begin{bmatrix}1 & -1\end{bmatrix}^\top$.

Similarly, for $\lambda_1 = \alpha+\beta$, $\beta \neq 0$, we solve $Bx = (\alpha+\beta)x$, $\beta \neq 0$,

$$\begin{aligned} \alpha x_0 + \beta x_1 &= (\alpha+\beta)x_0, \\ \beta x_0 + \alpha x_1 &= (\alpha+\beta)x_1, \end{aligned}$$

yielding $x_0 = x_1$. The eigenvector associated with λ_1 is thus $v_1 = 1/\sqrt{2}\begin{bmatrix}1 & 1\end{bmatrix}^\top$.

If $\beta = 0$ (and $\alpha \neq 0$), then B has a single eigenvalue $\lambda = 1$ with multiplicity 2 and two associated eigenvectors: $\begin{bmatrix}1 & 0\end{bmatrix}^\top$ and $\begin{bmatrix}0 & 1\end{bmatrix}^\top$.

(ii) $V = \begin{bmatrix}v_0 & v_1\end{bmatrix}$, and thus

$$V\Lambda V^\top \;=\; \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \;=\; \tfrac{1}{2}\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \;=\; A.$$

It is also true that

$$V\begin{bmatrix} \alpha-\beta & 0 \\ 0 & \alpha+\beta \end{bmatrix}V^\top \;=\; B.$$

V is a unitary matrix since $VV^\top = V^\top V = I$; it is formed from two orthonormal vectors.

(iii) Knowing the eigenvalues, it is easy to compute the determinant as the product of those eigenvalues:

$$\det(A) \;=\; -3.$$

A is thus an invertible matrix, and its inverse is

$$A^{-1} \;=\; -\tfrac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

This inverse could have been computed easily as

$$\begin{aligned} A^{-1} \;&=\; (V\Lambda V^\top)^{-1} \;=\; (V^\top)^{-1}\Lambda^{-1}V^{-1} \;=\; V\Lambda^{-1}V^\top \\ &=\; \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

(iv) We again compute the determinant as the product of the eigenvalues,

$$\det(B) \;=\; (\alpha-\beta)(\alpha+\beta).$$

B is not invertible if and only if $\alpha = \beta$ or $\alpha = -\beta$. The inverse is

$$B^{-1} \;=\; \frac{1}{(\alpha-\beta)(\alpha+\beta)}\begin{bmatrix} \alpha & -\beta \\ -\beta & \alpha \end{bmatrix}, \qquad \alpha \neq \pm\beta.$$

As a sanity check, when $\alpha = 1$, $\beta = 2$, then $B = A$ and $B^{-1} = A^{-1}$.

2.56. *Operator norm, singular values, and eigenvalues*

(i) If the matrix A is Hermitian, that is, $A = A^*$, then $AA^* = A^2$, and the eigenvalues of A and A^* coincide. Furthermore, if λ_k is an eigenvalue of A that corresponds to an eigenvector v_k, then

$$AA^*v_k \;=\; A(\lambda_k v_k) \;=\; \lambda_k(Av_k) \;=\; \lambda_k^2 v_k.$$

Hence, λ_k^2 is an eigenvalue of AA^*, and by definition, $\sigma_k = \sqrt{\lambda_k^2} = |\lambda_k|$ is a singular value of A.

(ii) This result follows immediately from (i) and (iii).

(iii) Let the singular value decomposition of the matrix A be $U\Sigma V^*$, where U and V are unitary matrices, and $\Sigma = \operatorname{diag}(\sigma_0, \sigma_1, \ldots)$ is a diagonal matrix of the singular values of A such that $\sigma_0 \geq \sigma_1 \geq \ldots$. Then

$$\begin{aligned} \|A\|_2 \;&=\; \sup_{\|x\|=1}\|Ax\| = \sup_{\|x\|=1}\|U\Sigma V^*x\| \;\overset{(a)}{=}\; \sup_{\|y\|=1}\|\Sigma y\| \\ &=\; \sup_{\|y\|=1}\sqrt{\sum|\sigma_k y_k|^2} \;=\; \sigma_0 \sup_{\|y\|=1}\sqrt{\sum\left|\frac{\sigma_k}{\sigma_0}y_k\right|^2} \\ &\leq\; \sigma_0 \sup_{\|y\|=1}\sqrt{\sum|y_k|^2} \;=\; \sigma_0 \sup_{\|y\|=1}\|y\| = \sigma_0, \end{aligned}$$

where (a) follows from the fact that multiplication by a unitary matrix does not change the norm. The upper bound σ_0 can be achieved with $y = \begin{bmatrix} 1 & 0 & 0 & \ldots \end{bmatrix}^\top$. Hence, $\|A\| = \sigma_0 = \max\{\sigma_k\}$.

2.57. *Least-squares solution to a system of linear equations*

(i) If y belongs to the range (column space) of A, then there exists x such that $Ax = y$,

$$\widehat{y} \;=\; A\widehat{x} \;\overset{(a)}{=}\; A(A^\top A)^{-1}A^\top y \;\overset{(b)}{=}\; A(A^\top A)^{-1}(A^\top A)\,x \;=\; Ax \;=\; y,$$

where (a) follows from (2.225b); and (b) from (2.224).

(ii) If y is orthogonal to the column space of A, then $y \in \mathcal{N}(A^\top)$, the null space of $A^\top$. That is, $A^\top y = 0$, and so

$$\widehat{y} \;=\; A(A^\top A)^{-1}A^\top y \;=\; 0.$$

(iii) Let $A \in \mathbb{R}^{M\times N}$ with $M > N$. The problem is that of "solving" $y = Ax$ in the sense of finding $\widehat{x}$ to minimize $\|y - A\widehat{x}\|$. We found the solution (2.225b) and (2.225c) by starting from

$$A^\top(y - A\widehat{x}) \;=\; 0. \tag{S2.57-1}$$

Considering (S2.57-1) elementwise gives

$$a_i^\top(y - A\widehat{x}) \;=\; 0, \qquad \text{for } i = 0,\, 1,\, \ldots,\, N-1, \tag{S2.57-2}$$

where a_i is the ith column of A.

Now let $\varepsilon = \|y - \widehat{y}\|^2 = (y - A\widehat{x})^\top(y - A\widehat{x})$. Then

$$\frac{\partial\varepsilon}{\partial\widehat{x}_i} \;=\; a_i^\top(y - A\widehat{x}) + (y - A\widehat{x})^\top a_i \;=\; 2a_i^\top(y - A\widehat{x}) \;\overset{(a)}{=}\; 0,$$

where (a) follows from (S2.57-2).

2.58. *Power of a matrix*
Since A is full rank, it is diagonalizable. Define $V = \begin{bmatrix} v_0 & v_1 & \ldots & v_{N-1}\end{bmatrix}$ as the matrix containing the eigenvectors v_i of A, and define $\Lambda = \operatorname{diag}(\lambda_0,\, \lambda_1,\, \ldots,\, \lambda_{N-1})$, a diagonal matrix containing the eigenvalues of A. We can thus write

$$\begin{aligned} A^k \;&\overset{(a)}{=}\; (V\Lambda V^{-1})^k \;=\; (V\Lambda V^{-1})(V\Lambda V^{-1})\cdots(V\Lambda V^{-1}) \\ &=\; V\Lambda(V^{-1}V)\Lambda\cdots(V^{-1}V)\Lambda V^{-1} \;=\; V\Lambda^k V^{-1}, \end{aligned}$$

where (a) follows from (2.227a); and $\Lambda^k = \operatorname{diag}(\lambda_0^k,\, \lambda_1^k,\, \ldots,\, \lambda_{N-1}^k)$.

2.59. *Properties of jointly Gaussian vectors*

(i) The more general definition of a jointly Gaussian random variable simplifies this problem greatly. Let $\mathrm{w} = A\mathrm{x}$. Any linear combination of components of w is a linear combination of components of x and hence is a Gaussian random variable. Thus, w is a jointly Gaussian random vector. The expressions for the mean and the covariance matrix follow from the linearity of the expectation:

$$\begin{aligned} \mu_\mathrm{w} \;&=\; \mathrm{E}[\,\mathrm{w}\,] \;=\; \mathrm{E}[\,A\mathrm{x}\,] \;=\; A\,\mathrm{E}[\,\mathrm{x}\,] \;=\; A\mu_\mathrm{x}, \\ \Sigma_\mathrm{w} \;&=\; \mathrm{E}\Big[\,(\mathrm{w}-\mu_\mathrm{w})(\mathrm{w}-\mu_\mathrm{w})^\top\,\Big] \;=\; \mathrm{E}\Big[\,(A\mathrm{x}-A\mu_\mathrm{x})(A\mathrm{x}-A\mu_\mathrm{x})^\top\,\Big] \\ &=\; \mathrm{E}\Big[\,A(\mathrm{x}-\mu_\mathrm{x})(\mathrm{x}-\mu_\mathrm{x})^\top A^\top\,\Big] \;=\; A\,\mathrm{E}\Big[\,(\mathrm{x}-\mu_\mathrm{x})(\mathrm{x}-\mu_\mathrm{x})^\top\,\Big]A^\top \;=\; A\Sigma_\mathrm{x}A^\top. \end{aligned}$$

(ii) The definition of a jointly Gaussian vector x requires the covariance matrix Σ_x to be symmetric and positive semidefinite. (This extends to any random vector since $(\mathrm{x}-\mu_\mathrm{x})(\mathrm{x}-\mu_\mathrm{x})^\top$ is a quadratic form and the expectation is linear.) The symmetry condition $\Sigma_\mathrm{x} = \Sigma_\mathrm{x}^\top$,

$$\begin{bmatrix} \Sigma_\mathrm{y} & \Sigma_{\mathrm{y},\mathrm{z}} \\ \Sigma_{\mathrm{z},\mathrm{y}} & \Sigma_\mathrm{z} \end{bmatrix} \;=\; \Sigma_\mathrm{x} \;=\; \Sigma_\mathrm{x}^\top \;=\; \begin{bmatrix} \Sigma_\mathrm{y}^\top & \Sigma_{\mathrm{z},\mathrm{y}}^\top \\ \Sigma_{\mathrm{y},\mathrm{z}}^\top & \Sigma_\mathrm{z}^\top \end{bmatrix}$$

implies all the specified symmetries: $\Sigma_\mathrm{y} = \Sigma_\mathrm{y}^\top$, $\Sigma_\mathrm{z} = \Sigma_\mathrm{z}^\top$, and $\Sigma_{\mathrm{y},\mathrm{z}} = \Sigma_{\mathrm{z},\mathrm{y}}^\top$.

(iii) As in (i), y is jointly Gaussian because any linear combination of its entries is a linear combination of the entries of x. The mean and covariance matrix follow from (i) by choosing

$$A \;=\; \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

with dimensions matching the decomposition of x into y and z.

(iv) Since x has a joint PDF, its subvectors y and z have joint PDFs as well. Thus, we can show the result through an equality of PDFs. To simplify our expressions, we provide a solution only for the case of $\mu_\mathrm{x} = \mathbf{0}$.

Using (2.262), we want to show that the conditional PDF of y given $\mathrm{z} = t$ is

$$f_{\mathrm{y}|\mathrm{z}}(s\,|\,t) \;=\; c_1\,\exp\!\left(-\frac{1}{2}(s-\mu_{\mathrm{y}|\mathrm{z}})^\top\Sigma_{\mathrm{y}|\mathrm{z}}^{-1}(s-\mu_{\mathrm{y}|\mathrm{z}})\right), \tag{S2.59-1}$$

with

$$\mu_{y|z} \;=\; \mu_y + \Sigma_{y,z}\Sigma_z^{-1}(t - \mu_z) \qquad \text{and} \qquad \Sigma_{y|z} \;=\; \Sigma_y - \Sigma_{y,z}\Sigma_z^{-1}\Sigma_{z,y},$$

where the scalar constant c_1 could be written explicitly but is fixed implicitly by the normalization property of the conditional PDF. Applying the multidimensional analogue of (2.257a) with the joint PDF expression from (2.262) gives

$$\begin{aligned}
f_{y|z}(s\,|\,t) \;&=\; \frac{f_{y,z}(s,t)}{f_z(t)} \\
&=\; c_2\,\frac{\exp\left(-\frac{1}{2}\left(\begin{bmatrix} s \\ t\end{bmatrix} - \begin{bmatrix}\mu_y \\ \mu_z\end{bmatrix}\right)^{\top}\Sigma_x^{-1}\left(\begin{bmatrix} s \\ t\end{bmatrix} - \begin{bmatrix}\mu_y \\ \mu_z\end{bmatrix}\right)\right)}{\exp\left(-\frac{1}{2}(t-\mu_z)^{\top}\Sigma_z^{-1}(t-\mu_z)\right)} \qquad \text{(S2.59-2)} \\
&=\; c_2\,\exp\left(-\frac{1}{2}\left(\begin{bmatrix} s \\ t\end{bmatrix} - \begin{bmatrix}\mu_y \\ \mu_z\end{bmatrix}\right)^{\top}\Sigma_x^{-1}\left(\begin{bmatrix} s \\ t\end{bmatrix} - \begin{bmatrix}\mu_y \\ \mu_z\end{bmatrix}\right) + \frac{1}{2}(t-\mu_z)^{\top}\Sigma_z^{-1}(t-\mu_z)\right),
\end{aligned}$$

where again the scalar constant c_2 could be written explicitly but is fixed implicitly. We thus need for (S2.59-1) and (S2.59-2) to be equal. We can prove it by using the partitioned matrix inverse

$$\Sigma_x^{-1} \;=\; \begin{bmatrix}\Sigma_y & \Sigma_{y,z} \\ \Sigma_{z,y} & \Sigma_z\end{bmatrix}^{-1} \;=\; \begin{bmatrix}\Xi & -\Xi\Sigma_{y,z}\Sigma_y^{-1} \\ -\Sigma_z^{-1}\Sigma_{z,y}\Xi & \Sigma_z^{-1} + \Sigma_z^{-1}\Sigma_{z,y}\Xi\Sigma_{y,z}\Sigma_z^{-1}\end{bmatrix},$$

where $\Xi = (\Sigma_y - \Sigma_{y,z}\Sigma_z^{-1}\Sigma_{z,y})^{-1}$, and expanding the products.

2.60. *Bayesian linear MMSE estimation via the orthogonality principle*
The form of the optimal linear estimator of one vector from another was derived from the projection theorem in Section 2.4.4. Thus, we can derive the LMMSE estimator from (2.85).

Let

$$z \;=\; \begin{bmatrix} y \\ 1\end{bmatrix}$$

so that the desired estimator is a linear function of z:

$$\widehat{x} \;=\; Ay + b \;=\; \begin{bmatrix} A & b\end{bmatrix}\begin{bmatrix} y \\ 1\end{bmatrix} \;=\; Bz. \qquad \text{(S2.60-1)}$$

Using (2.85), the optimal B is given by

$$B \;=\; \mathrm{E}[\,xz^*\,]\,(\mathrm{E}[\,zz^*\,])^{-1}. \qquad \text{(S2.60-2)}$$

We need to find the factors in this expression.

The first factor is

$$\mathrm{E}[\,xz^*\,] \;=\; \mathrm{E}\big[\,x\begin{bmatrix} y^* & 1\end{bmatrix}\big] \;=\; \mathrm{E}\big[\begin{bmatrix} xy^* & x\end{bmatrix}\big] \;=\; \begin{bmatrix}\Sigma_{x,y} + \mu_x\mu_y^* & \mu_x\end{bmatrix}. \qquad \text{(S2.60-3)}$$

Since

$$\mathrm{E}[\,zz^*\,] \;=\; \mathrm{E}\left[\begin{bmatrix} y \\ 1\end{bmatrix}\begin{bmatrix} y^* & 1\end{bmatrix}\right] \;=\; \mathrm{E}\left[\begin{bmatrix} yy^* & y \\ y^* & 1\end{bmatrix}\right] \;=\; \begin{bmatrix}\Sigma_y + \mu_y\mu_y^* & \mu_y \\ \mu_y^* & 1\end{bmatrix},$$

matrix inversion gives

$$(\mathrm{E}[\,zz^*\,])^{-1} \;=\; \begin{bmatrix}\Sigma_y^{-1} & -\Sigma_y^{-1}\mu_y \\ -\mu_y^*\Sigma_y^{-1} & 1 + \mu_y^*\Sigma_y^{-1}\mu_y\end{bmatrix} \qquad \text{(S2.60-4)}$$

for the second factor. Substituting (S2.60-3) and (S2.60-4) into (S2.60-2) gives

$$B \;=\; \begin{bmatrix}\Sigma_{x,y}\Sigma_y^{-1} & \mu_x - \Sigma_{x,y}\Sigma_y^{-1}\mu_y\end{bmatrix}.$$

Substituting into (S2.60-1), we find that the desired LMMSE estimator is

$$\widehat{x} \;=\; \Sigma_{x,y}\Sigma_y^{-1}y + \mu_x - \Sigma_{x,y}\Sigma_y^{-1}\mu_y \;=\; \mu_x + \Sigma_{x,y}\Sigma_y^{-1}(y - \mu_y),$$

matching the result in (2.266a).

2.61. *An inner product on random vectors*

(i) Given the following assumptions

$$\langle \mathrm{x},\, \mathrm{y} \rangle \;=\; \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k \mathrm{y}_k^* \,], \qquad \mathrm{var}(\mathrm{x}_k) \leq \infty \quad \text{for all } k,$$

let us verify the inner product properties from Definition 2.7:

1. *Distributivity:*

$$\begin{aligned} \langle \mathrm{x}+\mathrm{y},\, \mathrm{z} \rangle \;&=\; \sum_{k=0}^{N-1} \mathrm{E}[\, (\mathrm{x}_k + \mathrm{y}_k)\, \mathrm{z}_k^* \,] \;=\; \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k \mathrm{z}_k^* \,] + \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{y}_k \mathrm{z}_k^* \,] \\ &=\; \langle \mathrm{x},\, \mathrm{z} \rangle + \langle \mathrm{y},\, \mathrm{z} \rangle. \end{aligned}$$

2. *Linearity in the first argument:*

$$\langle \alpha\mathrm{x},\, \mathrm{y} \rangle \;=\; \sum_{k=0}^{N-1} \mathrm{E}[\, \alpha \mathrm{x}_k \mathrm{y}_k^* \,] \;=\; \alpha \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k \mathrm{y}_k^* \,] \;=\; \alpha \langle \mathrm{x},\, \mathrm{y} \rangle.$$

3. *Hermitian symmetry:*

$$\langle \mathrm{x},\, \mathrm{y} \rangle^* \;=\; \left(\sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k \mathrm{y}_k^* \,] \right)^* \;=\; \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k^* \mathrm{y}_k \,] \;=\; \langle \mathrm{y},\, \mathrm{x} \rangle.$$

4. *Positive definiteness:*

$$\langle \mathrm{x},\, \mathrm{x} \rangle \;=\; \sum_{k=0}^{N-1} \mathrm{E}[\, \mathrm{x}_k \mathrm{x}_k^* \,] \;=\; \sum_{k=0}^{N-1} \mathrm{E}\big[\, |\mathrm{x}_k|^2 \,\big] \;\geq\; 0.$$

We have equality if and only if $\mathrm{E}\big[\, |\mathrm{x}_k|^2 \,\big] = 0$ for all k.

(ii) Two vectors x and y are orthogonal under this inner product if *on average* they are geometrically orthogonal. This does not mean, however, that all components are orthogonal.

(iii) As seen in (ii), all orthogonal vectors are not necessarily uncorrelated. However, uncorrelated vectors are orthogonal since

$$\langle \mathrm{x},\, \mathrm{y} \rangle \;=\; \sum_k \;=\; \mathrm{E}[\, \mathrm{x}_k \mathrm{y}_k^* \,] \;\stackrel{(a)}{=}\; \sum_k 0 \;=\; 0,$$

where (a) follows from the assumption.

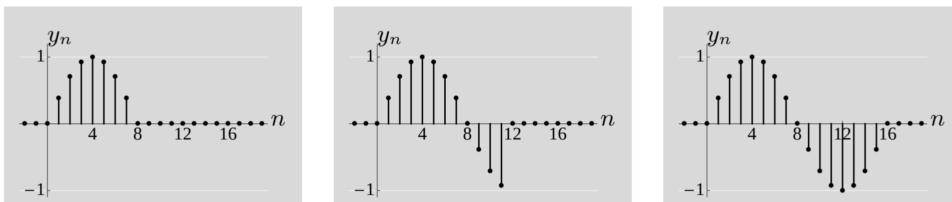
(iv) As we have seen, if the vectors are orthogonal, they are not necessarily uncorrelated, and thus not necessarily independent. Uncorrelated Gaussian random vectors are independent.

Chapter 3

Solutions to exercises

3.1. *Sinusoidal sequence*

(i) The three sketches are given in Figure S3.1-1.



(a) $N = 8$. (b) $N = 12$. (c) $N = 16$.

Figure S3.1-1 Sequences y from (P3.1-1).

(ii) This is true only for $N = 16$ because x is periodic with period 16, as shown in Figure S3.1-2.

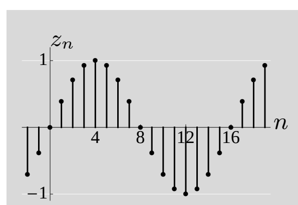


Figure S3.1-2 Sequence $z_n = \sum_{k\in\mathbb{Z}} y_{n-Nk}$. For $N = 16$, $z_n = x_n$.

3.2. *Deterministic autocorrelation and crosscorrelation*

(i) Use the definition of the deterministic autocorrelation (3.17) and write

$$a_n \;=\; \sum_{k\in\mathbb{Z}} x_k\, x_{k-n}^* \;\overset{(a)}{=}\; \left(\sum_{k\in\mathbb{Z}} x_{k-n}\, x_k^*\right)^* \;\overset{(b)}{=}\; \left(\sum_{m\in\mathbb{Z}} x_m\, x_{m+n}^*\right)^* \;=\; a_{-n}^*,$$

where (a) follows from conjugating the expression twice; and (b) from the change of variable $m = k + n$.

(ii) Fix any $n \in \mathbb{Z}$, and define sequence y by $y_k = x_{k-n}$ for all $k \in \mathbb{Z}$. Then, $\|y\| = \|x\|$, and

$$|a_n| \;=\; \left|\sum_{k\in\mathbb{Z}} x_k x_{k-n}^*\right| \;=\; \left|\sum_{k\in\mathbb{Z}} x_k y_k^*\right| \;=\; |\langle x,\, y\rangle| \;\overset{(a)}{\le}\; \|x\|\,\|y\| \;=\; \|x\|^2 \;\overset{(b)}{=}\; a_0,$$

where (a) follows from the Cauchy–Schwarz inequality; and (b) from (3.18b).

(iii) To show that the deterministic crosscorrelation is not symmetric, we give a counterexample. Let $x_n = \delta_n$ and $y_n = \delta_{n-1}$. Then

$$c_{x,y,-1} \;=\; \sum_{k\in\mathbb{Z}} x_k\, y_{k+1}^* \;=\; x_0\, y_1 \;=\; 1$$

but

$$c_{x,y,1} = \sum_{k\in\mathbb{Z}} x_k\, y^*_{k-1} = 0.$$

While the deterministic crosscorrelation is not symmetric in general, the following holds

$$c_{x,y,-1} = c_{y,x,1} = 1.$$

(iv) We write

$$\begin{aligned}
C(e^{j\omega}) &= \sum_{n\in\mathbb{Z}} c_n e^{-j\omega n} \overset{(a)}{=} \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_k y^*_{k-n} e^{-j\omega n} \\
&\overset{(b)}{=} \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_k y^*_m e^{-j\omega(k-m)} = \sum_{k\in\mathbb{Z}} x_k e^{-j\omega k}\left(\sum_{m\in\mathbb{Z}} y_m e^{-j\omega m}\right)^* \\
&= X(e^{j\omega})Y^*(e^{j\omega}),
\end{aligned}$$

where (a) follows from the definition of deterministic crosscorrelation (3.20); and (b) from the change of variable $m = k - n$.

3.3. *Discrete Laplacian operator*
Denote by T the operator that describes the system $y = T(x)$.

(i) T is linear because

$$\begin{aligned}
T(\alpha x_{0,n} + \beta x_{1,n}) &= (\alpha x_{0,n-1} + \beta x_{1,n-1}) - 2(\alpha x_{0,n} + \beta x_{1,n}) + (x_{0,n+1} + x_{1,n+1}) \\
&= \alpha(x_{0,n-1} - 2x_{0,n} + x_{0,n+1}) + \beta(x_{1,n-1} - 2x_{1,n} + x_{1,n+1}) \\
&= \alpha T(x_{0,n}) + \beta T(x_{1,n}).
\end{aligned}$$

T is shift-invariant because

$$T(x_{n-k}) = x_{n-k-1} - 2x_{n-k} + x_{n-k+1} = y_{n-k},$$

that is, a shifted input produces a shifted output.
T is not causal because y_n depends on x_{n+1}, which is a future value of x.
T is not memoryless because y_n does not depend only on x_n.
T is BIBO-stable because

$$|x_n| \le M \qquad \Rightarrow \qquad |y_n| \le 4M.$$

(ii) Because the system is linear, it has a matrix representation,

$$\begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & -2 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & -2 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \boxed{-2} & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -2 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & -2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}.$$

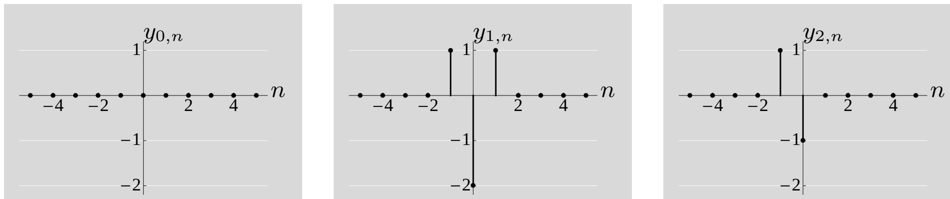
(iii) The system acts as a *discrete differentiator*, so constant and linear terms get annihilated (see Figure S3.3-1).

3.4. *Linear and shift-invariant difference equations*

(i) Consider the following difference equation:

$$y_n = x_n - y_{n-1},$$

with the initial condition $y_{-1} = 2$. Take the input signal to be $x_n = \delta_n$. The output signal is then $y_n = (-1)^{n+1}$, for $n \ge 0$. For $x'_n = 2x_n$, the output is $y'_n = 0 \ne 2y_n$; thus, the system is not linear. For $x'_n = x_{n-1}$ the output is $y'_0 = -2$, $y'_n = 3\,(-1)^{n+1}$. Since $y'_n \ne y_{n-1}$, the system is not shift-invariant.



(a) $x_{0,n} = c$. (b) $x_{1,n} = \delta_n$. (c) $x_{2,n} = u_n$.

Figure S3.3-1 The output of the system for different inputs x.

(ii) If initial conditions are zero, then:

(a) By induction, the solution of the homogeneous equation

$$y_n = -\sum_{k=1}^{N} a_k y_{n-k}$$

is $y_n = 0$.

(b) If input signals x'_n and x''_n produce outputs y'_n and y''_n, respectively, then $x_n = \alpha x'_n + \beta x''_n$ produces the output $y_n = \alpha y'_n + \beta y''_n$:

$$\begin{aligned} y_n &= \alpha y'_n + \beta y''_n \\ &= \alpha\left(\sum_{k=0}^{M} b_k x'_{n-k} - \sum_{k=1}^{N} a_k y'_{n-k}\right) + \beta\left(\sum_{k=0}^{M} b_k x''_{n-k} - \sum_{k=1}^{N} a_k y''_{n-k}\right) \\ &= \sum_{k=0}^{M} b_k(\alpha x'_{n-k} + \beta x''_{n-k}) - \sum_{k=1}^{N} a_k(\alpha y'_{n-k} + \beta y''_{n-k}) \\ &= \sum_{k=0}^{M} b_k x_{n-k} - \sum_{k=1}^{N} a_k y_{n-k}. \end{aligned}$$

Hence, the system is linear.

(c) If the input signals x_n produces the output y_n, then $x'_n = x_{n-m}$ produces the output $y'_n = y_{n-m}$:

$$\begin{aligned} y'_n = y_{n-m} &= \sum_{k=0}^{M} b_k x_{n-m-k} - \sum_{k=1}^{N} a_k y_{n-m-k} \\ &= \sum_{k=0}^{M} b_k x'_{n-k} - \sum_{k=1}^{N} a_k y'_{n-k}; \end{aligned}$$

thus, the system is shift-invariant.

3.5. *Geometric sequences and their properties*

(i) The norm of x is

$$\|x\|_2^2 \;=\; \sum_{n=0}^{\infty}\left(\sqrt{1-\alpha^2}\right)^2 \alpha^{2n} \;=\; (1-\alpha^2)\sum_{n=0}^{\infty}\alpha^{2n} \;\overset{(a)}{=}\; (1-\alpha^2)\,\frac{1}{1-\alpha^2} \;=\; 1,$$

where (a) follows from (P2.54-3).

(ii) For $n < 0$, the autocorrelation of x is

$$\begin{aligned} a_n &= \sum_{k\in\mathbb{Z}} x_k x_{k-n} \;=\; \sum_{k=0}^{\infty}(1-\alpha^2)\,\alpha^k\alpha^{k-n} \\ &= \alpha^{-n}\sum_{k=0}^{\infty}(1-\alpha^2)\,\alpha^{2k} \;=\; \alpha^{-n}\|x\|_2^2 \;=\; \alpha^{-n}; \end{aligned}$$

and for $n \geq 0$,

$$\begin{aligned} a_n &= \sum_{k\in\mathbb{Z}} x_k x_{k-n} = \sum_{k=n}^{\infty} (1-\alpha^2)\,\alpha^k \alpha^{k-n} = \alpha^{-n} \sum_{k=n}^{\infty} (1-\alpha^2)\,\alpha^{2k} \\ &\stackrel{(a)}{=} \alpha^n \sum_{\ell=0}^{\infty} (1-\alpha^2)\,\alpha^{2\ell} = \alpha^n \|x\|_2^2 = \alpha^n, \end{aligned}$$

where (a) follows from change of variable $\ell = k - n$. We thus get

$$a_n = \alpha^{|n|};$$

the autocorrelation of a geometric sequence is a symmetric, two-sided geometric sequence.

(iii) For $n < 0$, the convolution of x with itself is

$$(x * x)_n = \sum_{k\in\mathbb{Z}} x_k x_{n-k} = 0;$$

and for $n \geq 0$,

$$(x * x)_n = \sum_{k\in\mathbb{Z}} x_k x_{n-k} = (1-\alpha^2) \sum_{k=0}^{n} \alpha^k \alpha^{n-k} = (1-\alpha^2)\alpha^n (n+1).$$

(iv) We define two geometric series, characterized by parameters α and β, with $|\alpha| < 1$ and $|\beta| < 1$.

For $n < 0$, the crosscorrelation of x and y is

$$\begin{aligned} c_n &= \sum_{k\in\mathbb{Z}} x_k y_{k-n} = \sum_{k=0}^{\infty} \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\,\alpha^k \beta^{k-n} \\ &= \beta^{-n} \sqrt{(1-\alpha^2)(1-\beta^2)} \sum_{k=0}^{\infty} (\alpha\beta)^k = \frac{\sqrt{(1-\alpha^2)(1-\beta^2)}}{1-\alpha\beta}\,\beta^{-n}; \end{aligned}$$

and for $n \geq 0$,

$$\begin{aligned} c_n &= \sum_{k\in\mathbb{Z}} x_k y_{k-n} = \sum_{k=m}^{\infty} \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\,\alpha^k \beta^{k-n} \\ &= \beta^{-n} \sqrt{(1-\alpha^2)(1-\beta^2)} \sum_{k=m}^{\infty} (\alpha\beta)^k \\ &\stackrel{(a)}{=} \alpha^n \sqrt{(1-\alpha^2)(1-\beta^2)} \sum_{\ell=0}^{\infty} (\alpha\beta)^\ell = \frac{\sqrt{(1-\alpha^2)(1-\beta^2)}}{1-\alpha\beta}\,\alpha^n, \end{aligned}$$

where (a) follows from change of variable $\ell = k - n$. We thus get

$$c_n = \begin{cases} \gamma\,\beta^{-n}, & n < 0; \\ \gamma\,\alpha^n, & n \geq 0, \end{cases}$$

with $\gamma = \sqrt{(1-\alpha^2)(1-\beta^2)}/(1-\alpha\beta)$; the crosscorrelation of two geometric sequences is an asymmetric, two-sided geometric sequence. Note that for $\alpha = \beta$ we obtain the previous result.

For $n < 0$, the convolution of x and y is

$$(x * y)_n = \sum_{k\in\mathbb{Z}} x_k y_{n-k} = 0;$$

and for $n \geq 0$,

$$\begin{aligned} (x * y)_n &= \sum_{k\in\mathbb{Z}} x_k y_{n-k} = \sqrt{(1-\alpha^2)(1-\beta^2)} \sum_{k=0}^{n} \alpha^k \beta^{n-k} \\ &= \beta^n \sqrt{(1-\alpha^2)(1-\beta^2)} \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \\ &= \sqrt{(1-\alpha^2)(1-\beta^2)}\,\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}, \end{aligned}$$

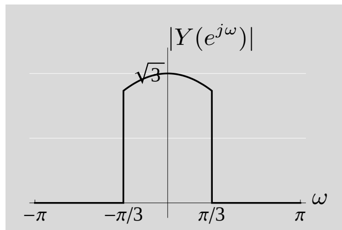


Figure S3.7-1 The DTFT of the output of the system $y = h*x$, with $x_n = \frac{1}{2}(\delta_n + \delta_{n-1})$ and h_n a third-band lowpass filter.

where (a) follows from the formula for the finite geometric series, (P2.54-1).

3.6. *Circular convolution in frequency property of the DTFT*
Let $y = h\, x$. Then, its DTFT is

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n\in\mathbb{Z}} y_n e^{-j\omega n} = \sum_{n\in\mathbb{Z}} x_n h_n\, e^{-j\omega n} \\
&= \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\nu})\, e^{j\nu n}\, d\nu\right) h_n\, e^{-j\omega n} \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} X(e^{j\nu})\, h_n\, e^{-j(\omega-\nu)n}\, d\nu \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\nu}) \left(\sum_{n\in\mathbb{Z}} h_n\, e^{-j(\omega-\nu)n}\right) d\nu \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\nu})\, H(e^{j(\omega-\nu)})\, d\nu = \frac{1}{2\pi} H \circledast X,
\end{aligned}$$

where we used the fact that $h \in \ell^1(\mathbb{Z})$.

3.7. *Third-band filter*

(i) We can rewrite h_n as

$$h_n = \sqrt{\frac{\omega_0}{2\pi}}\, \mathrm{sinc}\left(\frac{1}{2}\omega_0 n\right), \qquad \omega_0 = \frac{2\pi}{3}.$$

Thus, from Tables 3.4 or 3.5,

$$H\left(e^{j\omega}\right) = \begin{cases} \sqrt{3}, & \text{for } |\omega| \le \frac{1}{3}\pi; \\ 0, & \text{otherwise.} \end{cases}$$

The filter is lowpass because it lets through low frequencies $|\omega| \le \frac{1}{3}\pi$ and blocks all others.

(ii) Using the convolution property of the DTFT,

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{2}\left(1 + e^{-j\omega}\right), \\
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) = \begin{cases} (\sqrt{3}/2)\left(1 + e^{-j\omega}\right), & \text{for } |\omega| \le \frac{1}{3}\pi; \\ 0, & \text{otherwise;} \end{cases} \\
|Y(e^{j\omega})| &= \begin{cases} \sqrt{3/2}\sqrt{1+\cos(\omega)}, & \text{for } |\omega| \le \frac{1}{3}\pi; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Figure S3.7-1 shows the plot of $|Y(e^{j\omega})|$.

3.8. *ROC of z-transform*

(i) For $x_n = \delta_n$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} \delta_n \, z^{-n} \;=\; z^0 \;=\; 1; \\ \text{ROC} &= \{z \mid z \in \mathbb{C}\}. \end{aligned}$$

(ii) For $x_n = \delta_{n-k}$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} \delta_{n-k} \, z^{-n} \;=\; z^{-k}; \\ \text{ROC} &= \begin{cases} \{z \mid z \in \mathbb{C}\}, & \text{for } k \le 0; \\ \{z \mid |z| \neq 0\}, & \text{for } k > 0. \end{cases} \end{aligned}$$

(iii) For $x_n = \alpha^n \, u_n$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} \alpha^n \, u_n \, z^{-n} \;=\; \sum_{n=0}^{\infty} (\alpha \, z^{-1})^n \;=\; \frac{1}{1-\alpha \, z^{-1}}; \\ \text{ROC} &= \{z \mid |z| > |\alpha|\}. \end{aligned}$$

(iv) For $x_n = -\alpha^n \, u_{-n-1}$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} -\alpha^n \, u_{-n-1} \, z^{-n} \;=\; -\sum_{n=-\infty}^{-1} (\alpha \, z^{-1})^n \;\overset{(a)}{=}\; -\sum_{m=1}^{\infty} (\alpha^{-1} \, z)^m \\ &\overset{(b)}{=}\; -\sum_{m=0}^{\infty} (\alpha^{-1} \, z)^m + 1 \;=\; -\frac{1}{1-\alpha^{-1} \, z} + 1 \;=\; \frac{1}{1-\alpha \, z^{-1}}; \\ \text{ROC} &= \{z \mid |z| < |\alpha|\}, \end{aligned}$$

where (a) follows from the change of variable $m = -n$; and in (b) we added and subtracted the zeroth term to be able to apply (P2.54-2).

(v) For $x_n = n\alpha^n \, u_n$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} n\alpha^n \, u_n \, z^{-n} \;=\; \sum_{n=0}^{\infty} n \, (\alpha \, z^{-1})^n \;=\; \alpha \, z^{-1} \sum_{n=0}^{\infty} n \, (\alpha \, z^{-1})^{n-1} \\ &\overset{(a)}{=}\; \alpha \, z^{-1} \, \frac{d(\sum_{n=0}^{\infty} (\alpha \, z^{-1})^n)}{d(\alpha \, z^{-1})} \;\overset{(b)}{=}\; \alpha \, z^{-1} \, \frac{d(1/1-\alpha \, z^{-1})}{d(\alpha \, z^{-1})} \;=\; \frac{\alpha \, z^{-1}}{(1-\alpha \, z^{-1})^2}; \\ \text{ROC} &= \{z \mid |z| > |\alpha|\}, \end{aligned}$$

where (a) follows from recognizing $\sum_{n=0}^{\infty} n \, (\alpha \, z^{-1})^{n-1}$ as the derivative of $\sum_{n=0}^{\infty} (\alpha \, z^{-1})^n$ with respect to $(\alpha \, z^{-1})$; and (b) from the solution to (iii).

(vi) For $x_n = -n\alpha^n \, u_{-n-1}$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} -n\alpha^n \, u_{-n-1} \, z^{-n} \;=\; -\sum_{n=-\infty}^{-1} n \, (\alpha \, z^{-1})^n \\ &= -\alpha \, z^{-1} \sum_{n=-\infty}^{-1} n \, (\alpha \, z^{-1})^{n-1} \\ &\overset{(a)}{=}\; \alpha \, z^{-1} \, \frac{d(-\sum_{n=-\infty}^{-1} (\alpha \, z^{-1})^n)}{d(\alpha \, z^{-1})} \\ &\overset{(b)}{=}\; \alpha \, z^{-1} \, \frac{d(1/1-\alpha \, z^{-1})}{d(\alpha \, z^{-1})} \;=\; \frac{\alpha \, z^{-1}}{(1-\alpha \, z^{-1})^2}; \\ \text{ROC} &= \{z \mid |z| < |\alpha|\}, \end{aligned}$$

where (a) follows from recognizing $\sum_{n=-\infty}^{-1} n \, (\alpha \, z^{-1})^{n-1}$ as the derivative of $\sum_{n=-\infty}^{-1} (\alpha \, z^{-1})^n$ with respect to $(\alpha \, z^{-1})$; and (b) from the solution to (iv).

(vii) For $x_n = \cos(\omega_0 n)\, u_n$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} \cos(\omega_0 n)\, u_n\, z^{-n} \;=\; \sum_{n=0}^{\infty} \cos(\omega_0 n)\, z^{-n} \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} (e^{j\omega_0 n} + e^{-j\omega_0 n})\, z^{-n} \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} (e^{j\omega_0} z^{-1})^n + \frac{1}{2}\sum_{n=1}^{\infty} (e^{-j\omega_0} z^{-1})^n \\ &= 1 + \frac{1}{2}\,\frac{e^{j\omega_0} z^{-1}}{1-e^{j\omega_0} z^{-1}} + \frac{1}{2}\,\frac{e^{-j\omega_0} z^{-1}}{1-e^{-j\omega_0} z^{-1}} \;=\; \frac{1-\cos\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}; \\ \text{ROC} &= \{z \mid |z| > 1\}. \end{aligned}$$

(viii) For $x_n = \sin(\omega_0 n)\, u_n$,

$$\begin{aligned} X(z) &= \sum_{n\in\mathbb{Z}} \sin(\omega_0 n)\, u_n\, z^{-n} \;=\; \sum_{n=0}^{\infty} \sin(\omega_0 n)\, z^{-n} \\ &= 1 + \frac{1}{2j}\sum_{n=1}^{\infty} (e^{j\omega_0 n} - e^{-j\omega_0 n})\, z^{-n} \\ &= 1 + \frac{1}{2j}\sum_{n=1}^{\infty} (e^{j\omega_0} z^{-1})^n - \frac{1}{2j}\sum_{n=1}^{\infty} (e^{-j\omega_0} z^{-1})^n \\ &= 1 + \frac{1}{2j}\,\frac{e^{j\omega_0} z^{-1}}{1-e^{j\omega_0} z^{-1}} - \frac{1}{2j}\,\frac{e^{-j\omega_0} z^{-1}}{1-e^{-j\omega_0} z^{-1}} \;=\; \frac{1-\sin\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}; \\ \text{ROC} &= \{z \mid |z| > 1\}. \end{aligned}$$

(ix) For $x_n = \alpha^n$ for $0 \le n \le N$, and 0 otherwise,

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} \alpha^n z^{-n} \;=\; \sum_{n=0}^{N-1} (\alpha z^{-1})^n \;=\; \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}}; \\ \text{ROC} &= \{z \mid z\in\mathbb{C}, z \neq 0\}. \end{aligned}$$

3.9. *Orthogonality*

(i) If $P(z)$ is a polynomial, then $P(z^{-1})$ is not, and thus, for (P3.9-1) to hold, $P(z)$ must be a monomial, $P(z) = \pm z^{-\ell}$.

(ii) The proposed solution satisfies the orthogonality constraint (P3.9-1),

$$P(z)P(z^{-1}) \;=\; \frac{A(z)}{\widetilde{A}(z)}\frac{A(z^{-1})}{\widetilde{A}(z^{-1})} \;=\; \frac{A(z)}{z^{-L+1}A(z^{-1})}\frac{A(z^{-1})}{z^{L-1}A(z)} \;=\; 1.$$

3.10. *Linear and circular convolution as polynomial products*

Call $C(z)$ the result of the multiplication

$$C(z) \;=\; A(z)B(z) \;=\; \sum_{n=0}^{2N-2} c_n z^n$$

since $A(z)$ and $B(z)$ are polynomials of degree $N-1$. The coefficients c_n can be found from the linear convolution of the sequences a and b,

$$\begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 \\ 0 & a_{N-1} & a_{N-2} & \cdots & a_1 \\ 0 & 0 & a_{N-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \\ c_N \\ \vdots \\ c_{2N-2} \end{bmatrix}. \qquad \text{(S3.10-1)}$$

With $D(z) = C(z) \bmod (z^N - 1)$, and because for $n = 0, 1, \ldots, N-1$, the coefficients $c_{N+n}\, z^{N+n} \bmod (z^N - 1) = c_{N+n}\, z^n$, we have that $d_n = c_n + c_{N+n}$, or,

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \end{bmatrix} = \begin{bmatrix} I_N & I_{N\times N-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \\ c_N \\ \vdots \\ c_{2N-2} \end{bmatrix}, \tag{S3.10-2}$$

where I_N is an identity matrix and $I_{N\times N-1}$ is a matrix with an identity matrix I_{N-1} followed by an all-0 row. Combining (S3.10-1) and (S3.10-2), we can write

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \end{bmatrix} = A \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \end{bmatrix},$$

with

$$\begin{aligned} A &= \begin{bmatrix} I_N & I_{N\times N-1} \end{bmatrix} \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 \\ 0 & a_{N-1} & a_{N-2} & \cdots & a_1 \\ 0 & 0 & a_{N-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} \end{bmatrix} \\ &= \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 \end{bmatrix} + \begin{bmatrix} 0 & a_{N-1} & a_{N-2} & \cdots & a_1 \\ 0 & 0 & a_{N-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \end{bmatrix} = \begin{bmatrix} a_0 & a_{N-1} & a_{N-2} & \cdots & a_1 \\ a_1 & a_0 & a_{N-1} & \cdots & a_2 \\ a_2 & a_1 & a_{N-0} & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \end{bmatrix},$$

exactly the representation for the circular convolution of sequences a and b.

3.11. *Deterministic autocorrelation*

(i) The deterministic autocorrelation is the convolution of the sequence with its time-reversed version, (3.62d),

$$a_n = \sum_{k\in\mathbb{Z}} x_k\, x_{k-n} = x_n *_n x_{-n}.$$

From Table 3.6, the z-transform of x_{-n} is $X(z^{-1})$, and thus, $A(z) = X(z)X(z^{-1})$.

If $X(z)$ has the ROC $\{z \mid m < |z| < M\}$, then $X(z^{-1})$ has the ROC $\{z \mid 1/M < |z| < 1/m\}$. The ROC of $A(z)$ is the intersection of these two ROCs, that is, $\{z \mid \max\{m, 1/M\} < |z| < \min\{M, 1/m\}\}$.

(ii) For x_n to be stable, $|\alpha| < 1$. From Table 3.6, we then have,

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad \text{for} \qquad |z| > |\alpha|,$$

and

$$X(z^{-1}) = \frac{1}{1-\alpha z} \qquad \text{for} \qquad |z| < |\alpha^{-1}|.$$

Therefore,

$$\begin{aligned} A(z) &= \frac{1}{1-\alpha z^{-1}} \frac{1}{1-\alpha z} \\ &= \frac{1}{1-\alpha^2}\left(\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\alpha^{-1}z^{-1}}\right) \qquad |\alpha| < |z| < |\alpha^{-1}|. \end{aligned}$$

From Table 3.6 we thus have $a_n = (\alpha^n u_n + \alpha^{-n} u_{-n-1})/(1-\alpha^2)$. For this deterministic autocorrelation sequence, the poles are α, α^{-1}, the zeros are $0, \infty$, and the ROC is $\{z \mid |\alpha| < |z| < |\alpha^{-1}|\}$.

(iii) Let $y_n = x_{-n}$, then $Y(z) = X(z^{-1})$, and thus $A_y(z) = Y(z)Y(z^{-1}) = X(z^{-1})X(z) = A(z)$, that is, their deterministic autocorrelations are the same. In other words, time reversal does not change deterministic autocorrelation.

(iv) Take $v_n = x_{n-n_0}$, then $V(z) = z^{-n_0}X(z)$, and thus $A_v(z) = z^{-n_0}X(z)\, z^{n_0}X(z^{-1}) = A(z)$, that is, their deterministic autorcorrelations are the same. In other words, shift in time does not change deterministic autocorrelation.

3.12. *Block circulant matrices*
A block-circulant matrix C has block $C_{(i-j) \bmod N}$ in its (i,j)th position. We want to show that the matrix

$$\Lambda = F\,C\,F^{-1}$$

is block diagonal. The (k,j)th block of Λ is given by

$$\begin{aligned} \Lambda_{k,j} &= \frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} W_N^{ki} C_{(i-j) \bmod N} W_N^{-j\ell} \\ &\overset{(a)}{=} \frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} W_N^{(k-\ell)i} C_{(i-j) \bmod N} W_N^{(i-j)\ell} \\ &\overset{(b)}{=} \frac{1}{N}\sum_{i=0}^{N-1}\sum_{m=0}^{N-1} W_N^{(k-\ell)i} C_m W_N^{m\ell} = \frac{1}{N}\sum_{i=0}^{N-1} W_N^{(k-\ell)i} \sum_{m=0}^{N-1} C_m W_N^{m\ell} \\ &\overset{(c)}{=} \delta_{k-\ell}\sum_m C_m W_N^{m\ell}, \end{aligned}$$

where in (a) we multiplied and divided by $W_N^{\ell i}$; (b) follows from the change of variable $m = (i-j) \bmod N$; and (c) from the orthogonality of the the roots of unity, (3.288c). From this we see that Λ is block diagonal.

3.13. *Pattern recognition*

(i) A sequence $x \in \mathbb{R}^N$ can be written as a linear combination of $\{\varphi_k\}_{K=0}^{N-1}$ only if they form a basis for $\mathbb{R}^N$. Because the basis sequences are circular shifts of p, the matrix Φ corresponding to $\{\varphi_k\}_{K=0}^{N-1}$ is circulant. We know from (3.181a) that the DFT diagonalizes the circular convolution operator

$$\Phi = F^{-1}\Lambda F,$$

where Λ is a diagonal matrix of DFT coefficients of p. Thus, Φ is full rank if and only if all these DFT coefficients are nonzero.

(ii) Assuming that the condition from (i) is satisfied, that is, φ_k, $k = 0, 1, \ldots, N-1$ form a basis for $\mathbb{R}^N$, we can expand x as

$$x = \Phi\alpha,$$

with α the vector of expansion coefficients. Since Φ is full rank, it is invertible, and thus

$$\alpha \;=\; \Phi^{-1}x \;=\; F^{-1}\Lambda^{-1}F\,x.$$

3.14. *Computing linear convolution with the DFT*
The equivalence of linear and circular convolution for $N \geq M+L-1$ has been shown in Theorem 3.10, and the computation of circular convolution using the DFT was shown in Section 3.6. These two results together answer the question.

The algorithm is as follows: Take $N \geq M+L-1$, and compute the DFTs of x and h extended to a length-N sequence each by appending zeroes. The result is

$$X \;=\; F\,x, \qquad H \;=\; F\,h,$$

with X and H vectors of DFT coefficients. From (3.171), the circular convolution in the time domain has the DFT pair

$$Y_k \;=\; H_k X_k, \qquad k = 0,\,1,\,\ldots,\,N-1.$$

Taking the inverse DFT,

$$y \;=\; F^{-1}\,Y.$$

Note that even if $N \gg M+L-1$ (that is, taking a much longer DFT than required), the sequence y will be zero for $n > M+L-1$, because of the equivalence between linear and circular convolutions.

3.15. *DFT properties*
Some of these DFT pairs are explicitly given in Table 3.7.

(i) For $y_n = x_{-n \bmod N} = x_{N-n}$, its DFT is

$$\begin{aligned}
Y_k \;&=\; \sum_{n=0}^{N-1} y_n W_N^{kn} \;=\; \sum_{n=0}^{N-1} x_{N-n} W_N^{kn} \;\stackrel{(a)}{=}\; \sum_{m=0}^{N-1} x_m W_N^{k(N-m)} \\
&=\; \sum_{m=0}^{N-1} x_m \underbrace{W_N^{kN}}_{=1} W_N^{-km} \;=\; \sum_{m=0}^{N-1} x_m W_N^{(-k)m} \;=\; \sum_{m=0}^{N-1} x_m W_N^{(N-k)m} \\
&=\; \sum_{m=0}^{N-1} x_m W_N^{(-k \bmod N)m} \;=\; X_{-k \bmod N},
\end{aligned}$$

where (a) follows from the change of variable $m = N-n$. For a real x, $Y_k = X_{-k} = X_k^*$ is also true.

(ii) For $y = h \circledast x$, we find the DFT pair from

$$\begin{aligned}
y_n \;&=\; (h \circledast x)_n \;=\; \sum_{n=0}^{N-1} h_n x_{(k-n) \bmod N} \\
&\stackrel{(a)}{=}\; \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{\ell=0}^{N-1} H_\ell W_N^{-\ell n} \sum_{m=0}^{N-1} X_m W_N^{-m(k-n)} \\
&=\; \frac{1}{N} \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} H_\ell X_m W_N^{-mk} \frac{1}{N} \sum_{n=0}^{N-1} W_N^{(m-\ell)n} \;\stackrel{(b)}{=}\; \frac{1}{N} \sum_{m=0}^{N-1} H_m X_m W_N^{-mk},
\end{aligned}$$

exactly the inverse DFT of $H\,X$. In the above, (a) follows from the inverse DFT (3.163b); and (b) from the orthogonality of the roots of unity (3.288c).

(iii) For $Y = (1/N)\,(H \circledast X)$, we find the DFT pair from

$$\begin{aligned}
Y_k \;&=\; \frac{1}{N}(H \circledast X)_k \;=\; \frac{1}{N} \sum_{n=0}^{N-1} H_n X_{(k-n) \bmod N} \\
&\stackrel{(a)}{=}\; \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\ell=0}^{N-1} h_\ell W_N^{n\ell} \sum_{m=0}^{N-1} x_m W_N^{(k-n)m} \\
&=\; \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} h_\ell x_m W_N^{km} \frac{1}{N} \sum_{n=0}^{N-1} W_N^{n(\ell-m)} \;\stackrel{(b)}{=}\; \sum_{m=0}^{N-1} h_m x_m W_N^{km}.
\end{aligned}$$

exactly the DFT of $h\,x$. In the above, (a) follows from the DFT (3.163a); and (b) from the orthogonality of the roots of unity (3.288c).

(iv) When $x_n = x_{-n \bmod N} = x_{N-n}$, its DFT is

$$\begin{aligned} X_k &= \tfrac{1}{2}(X_k + X_k) \\ &= \frac{1}{2}\left(\sum_{n=0}^{N-1} x_n W_N^{kn} + \sum_{n=0}^{N-1} x_{N-n} W_N^{kn}\right) = \frac{1}{2}\left(\sum_{n=0}^{N-1} x_n W_N^{kn} + \sum_{n=0}^{N-1} x_n W_N^{-kn}\right) \\ &= \sum_{n=0}^{N-1} x_n \tfrac{1}{2}(W_N^{kn} + W_N^{-kn}) = \sum_{n=0}^{N-1} x_n \cos\left(\frac{2\pi}{N}kn\right), \end{aligned}$$

and is real.

(v) When $x_n = -x_{-n \bmod N} = -x_{N-n}$, its DFT is

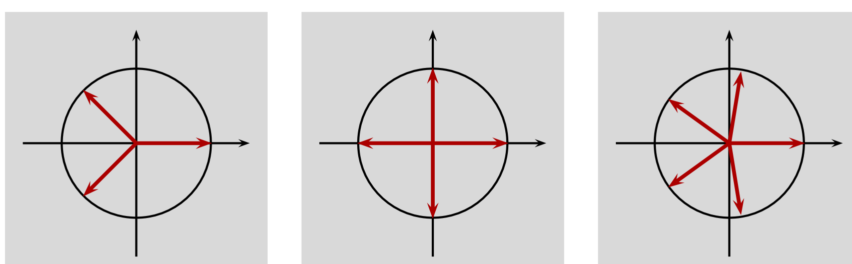
$$\begin{aligned} X_k &= \tfrac{1}{2}(X_k + X_k) \\ &= \frac{1}{2}\left(\sum_{n=0}^{N-1} x_n W_N^{kn} - \sum_{n=0}^{N-1} x_{N-n} W_N^{kn}\right) = \frac{1}{2}\left(\sum_{n=0}^{N-1} x_n W_N^{kn} - \sum_{n=0}^{N-1} x_n W_N^{-kn}\right) \\ &= \sum_{n=0}^{N-1} x_n \tfrac{1}{2}(W_N^{kn} - W_N^{-kn}) = j\sum_{n=0}^{N-1} x_n \sin\left(\frac{2\pi}{N}kn\right), \end{aligned}$$

and is purely imaginary.

3.16. *Tight frames as projections from orthonormal bases*

(i) Figure S3.16-1 shows the frame vectors for $N = 3, 4, 5$.

(a) For $N = 3$, consecutive frame vectors are separated by $2\pi/3$.
(b) For $N = 4$, consecutive frame vectors are separated by $2\pi/4$.
(c) For $N = 5$, consecutive frame vectors are separated by $2\pi/5$.



(a) $N = 3$. (b) $N = 4$. (c) $N = 5$.

Figure S3.16-1 Frame vectors.

(ii) Call r_k the column vectors of the DFT matrix. We know these vectors are orthogonal, but not orthonormal,

$$\langle r_k, r_n \rangle = N\delta_{k-n}.$$

Since these column vectors are the row vectors of Φ normalized by $1/\sqrt{M}$,

$$\left\langle \frac{1}{\sqrt{M}}r_k, \frac{1}{\sqrt{M}}r_n \right\rangle = \frac{1}{M}\langle r_k, r_n\rangle = \frac{N}{M}\delta_{k-n}.$$

The result then follows directly.

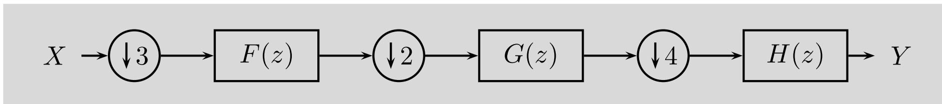


Figure S3.19-1 Multirate system.

3.17. *Downsampling by N*

We have

$$\begin{aligned} Y(z) &= \sum_{k\in\mathbb{Z}} x_{kN} z^{-k} = \sum_{m=kN} x_m z^{-m/N} \overset{(a)}{=} \sum_{m\in\mathbb{Z}} x_m z^{-m/N} \left(\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-nm}\right) \\ &= \frac{1}{N}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{N-1} x_m \left(W_N^n z^{1/N}\right)^{-m} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m\in\mathbb{Z}} x_m \left(W_N^n z^{1/N}\right)^{-m} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} X\left(W_N^n z^{1/N}\right), \end{aligned}$$

where (a) follows from the orthogonality of the roots of unity, (3.288c).

3.18. *Downsampling*

(i) For $k = 0, 1, \ldots, N/2-1$,

$$\begin{aligned} \frac{1}{2}\left(X_k + X_{k+N/2}\right) &= \frac{1}{2}\left(\sum_{n=0}^{N-1} x_n W_N^{kn} + \sum_{n=0}^{N-1} x_n W_N^{(k+N/2)n}\right) \\ &\overset{(a)}{=} \frac{1}{2}\sum_{n=0}^{N-1}\left(1+(-1)^n\right) x_n W_N^{kn} \\ &\overset{(b)}{=} \sum_{\ell=0}^{N/2-1} x_{2\ell} W_N^{k2\ell} \overset{(c)}{=} \sum_{\ell=0}^{N/2-1} y_\ell W_{N/2}^{k\ell} = Y_k, \end{aligned}$$

where (a) follows from $W_N^{N/2} = -1$; (b) from (3.288c); and (c) from $W_N^2 = W_{N/2}$.

(ii) For $k = 0, 1, \ldots, N/M-1$,

$$\begin{aligned} \frac{1}{M}\sum_{i=0}^{M-1} X_{k+iN/M} &= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{n=0}^{N-1} x_n W_N^{(k+iN/M)n} \\ &= \frac{1}{M}\sum_{n=0}^{N-1}\left(\sum_{i=0}^{M-1} W_N^{iN/Mn}\right) x_n W_N^{kn} \\ &\overset{(a)}{=} \frac{1}{M}\sum_{n=0}^{N-1}\left(\sum_{i=0}^{M-1} W_M^{in}\right) x_n W_N^{kn} \\ &\overset{(b)}{=} \sum_{\ell=0}^{N/M-1} x_{M\ell} W_N^{kM\ell} \overset{(c)}{=} \sum_{\ell=0}^{N/M-1} y_\ell W_{N/M}^{k\ell} = Y_k, \end{aligned}$$

where (a) follows from $W_N^{N/M} = W_M$; (b) from (3.288c); and (c) from $W_N^M = W_{N/M}$.

3.19. *Multirate system with different sampling rates*

(i) See Figure S3.19-1.

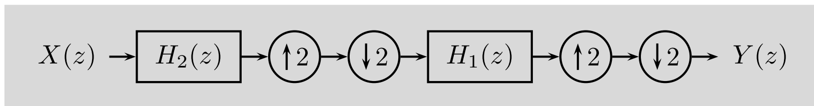


Figure S3.20-1 Equivalent multirate system.

(ii) The equivalent filter is

$$F(z^3)G(z^6)H(z^{24}),$$

followed by downsampling by 24,

$$Y(z) = \frac{1}{24}\sum_{k=0}^{23} F(W_{24}^{-k}z^{1/8})G(W_{24}^{-k}z^{1/4})H(W_{24}^{-k}z)\,X(W_{24}^{-k}z^{1/24}).$$

3.20. *Multirate identities*

(i) Using the fact that filtering followed by upsampling is equivalent to upsampling followed by upsampled filtering, we move both H_2 and H_1 across upsamplers to get the system in Figure S3.20-1. We also know that upsampling followed by downsampling by the same factor is identity. Therefore the transfer function of this system is:

$$\frac{Y(z)}{X(z)} = H_1(z)H_2(z).$$

(ii) Using again the interchange of filtering and upsampling, we can redraw the system as in Figure S3.20-2. The lower branch contains an upsampler followed by a delay and a downsampler. The output of such a system is 0. Therefore only the upper branch remains and the final transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H_0(z).$$

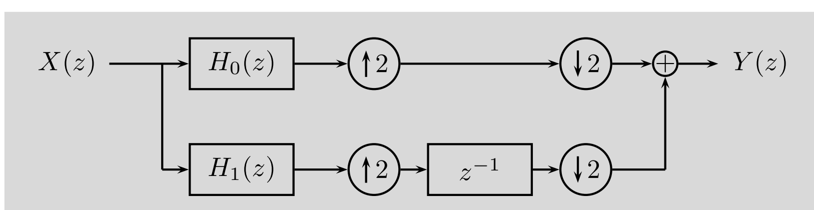


Figure S3.20-2 Equivalent multirate system.

(iii) For the first system, the input/output relationship is

$$\frac{Y(z)}{X(z)} = \frac{1}{2}\left[H(z^{1/2})G(z^{1/2}) + H(-z^{1/2})G(-z^{1/2})\right] \overset{(a)}{=} 1,$$

where (a) follows from (P3.20-1a).

For the second system, the input/output relationship is

$$\frac{Y(z)}{X(z)} = \frac{1}{2}\left[H(z^{1/2})F(z^{1/2}) + H(-z^{1/2})F(-z^{1/2})\right] \overset{(a)}{=} 0,$$

where (a) follows from (P3.20-1b).

3.21. *Interchange of multirate operations and filtering*

(i) Using multirate identities, $D_n = D_8$ and $H(z) = A(z)A(z^2)A(z^4)$ to yield $y = D_8 H x$.

(ii) $H(z)$ is an ideal lowpass filter, see Figure S3.21-1.

(iii) $H(z)$ is an ideal bandpass filter, see Figure S3.21-1. Despite A being an ideal highpass filter, the highest frequencies will get filtered out in the second iteration of the filter-downsample block, and thus, this system will not keep the highest frequency content of the input.

3.22. *Commutativity of upsampling and downsampling*
In (3.186a), D_2 is the downsampling-by-2 operator, an identity matrix with odd rows taken out. Similarly, in (3.192a), U_2 is the upsampling-by-2 operator, an identity matrix with zero rows inserted between every two rows. Similarly, D_M is the downsampling-by-M operator, an identity matrix with rows $kM + j$, $j = 1, 2, \ldots, M-1$, taken out, and U_N is the upsampling-by-N operator, an identity matrix with $(N-1)$ zero rows inserted between every two rows. Then,

$$(D_M U_N)_{ij} \;=\; \begin{cases} 1, & \text{for } Mi = Nj; \\ 0, & \text{otherwise,} \end{cases} \qquad (U_N D_M)_{ij} \;=\; \begin{cases} 1, & \text{for } i/N = j/M \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

For the above to hold, it must hold element-by-element, and thus, $Mi = Nj$, or $i/N = j/M = k \in \mathbb{Z}$ must hold for all $i, j \in \mathbb{Z}$. We now show that this is possible if and only if $\gcd(M, N) = 1$.

If $\gcd(M, N) = 1$, then

$$\frac{i}{N} \;=\; \frac{j}{M} \;=\; k \qquad \Rightarrow \qquad Mi \;=\; Nj \;=\; MNk,$$

and

$$Mi \;=\; Nj \qquad \Rightarrow \qquad i \;=\; Nq, \; j \;=\; Mq \qquad \Rightarrow \qquad \frac{i}{N} \;=\; q \;=\; \frac{j}{M}.$$

We prove necessity by contradiction. Let $\gcd(M, N) = L$. Then, $M = M'L$ and $N = N'L$, where $\gcd(M', N') = 1$. Further, $Mi = Nj$ implies that $i = N'q$, $j = M'q$, and $i/N = q/L = j/M = k \in \mathbb{Z}$ must hold for any $q \in \mathbb{Z}$, possible only for $L = 1$ and $q = k$. Hence, $L = \gcd(M, N) = 1$.

3.23. *Combinations of upsampling and downsampling*
We solve the problem using matrix notation for U_3, U_4 and D_2:

$$U_3 \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}, \qquad U_4 \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix},$$

and

$$D_2 \;=\; \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

(i) For the first comparison, compute

$$U_3D_2 = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad D_2U_3 = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Thus $U_3D_2\,x$ and $D_2U_3\,x$ are identical since $U_3D_2 = D_2U_3$.

(ii) For the second comparison, compute

$$U_4D_2 = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad D_2U_4 = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

which are evidently not the same, and thus $U_4D_2\,x \neq D_2U_4\,x$.

The results of the two comparisons are different because in the first case, 3 and 2 are coprime and thus D_2 and U_3 commute, while in the second, 4 and 2 are not coprime and thus D_2 and U_4 do not commute (see Exercise 3.22).

3.24. *Interchange of filtering and sampling rate change*

(i) Denote the input to the system by x. Using (3.188), the z-transform of the sequence after downsampling by 2 is

$$\frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right].$$

Filtering with $\widetilde{G}(z)$ then results in a sequence with z-transform

$$\frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right]\widetilde{G}(z).$$

Alternatively, first filtering with $\widetilde{G}(z^2)$ results in a sequence with z-transform $X(z)\widetilde{G}(z^2)$. Using (3.188), downsampling by 2 now results in a sequence with z-transform

$$\begin{aligned}&\frac{1}{2}\left[X(z^{1/2})\widetilde{G}((z^{1/2})^2) + X(-z^{1/2})\widetilde{G}((-z^{1/2})^2)\right] \\ &= \frac{1}{2}\left[X(z^{1/2})\widetilde{G}(z) + X(-z^{1/2})\widetilde{G}(z)\right] \;=\; \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right]\widetilde{G}(z),\end{aligned}$$

matching the previous expression.

(ii) Denote the input to the system by x. Filtering with $G(z)$ results in a sequence with z-transform $X(z)G(z)$. Using (3.193), upsampling by 2 then results in a sequence with z-transform

$$X(z^2)G(z^2).$$

Alternatively, using (3.193), first upsampling results in a sequence with z-transform $X(z^2)$. Filtering with $G(z^2)$ then results in a sequence with z-transform

$$X(z^2)G(z^2),$$

matching the previous expression.

3.25. *Periodically shift-varying systems*
A linear, shift-varying system is characterized by a two-variable impulse response $h_{k,n}$, the response of the system to the input $x_n = \delta_{n-k}$. Since the input can be written as $x_n = \sum_k x_k \delta_{n-k}$, the output is given by $y_n = \sum_k x_k h_{k,n}$.

When $h_{k,n}$ is periodic in k with period N, we define the polyphase components x_k as in (3.230), to yield

$$x_{k,n} = x_{nN+k}, \qquad x_k = \begin{bmatrix} \cdots & x_{-2N+k} & x_{-N+k} & \boxed{x_k} & x_{N+k} & x_{2N+k} & \cdots \end{bmatrix}^\top,$$

for $k \in \{0, 1, \ldots, N-1\}$. Denote the upsampled version of $x_{k,n}$ by $x_{k,n}^{(N)}$, so

$$\begin{aligned} x_{k,n}^{(N)} &= \begin{cases} x_{k,n/N}, & \text{for } n/N \in \mathbb{Z}; \\ 0, & \text{otherwise}; \end{cases} = \begin{cases} x_{n+k}, & \text{for } n/N \in \mathbb{Z}; \\ 0, & \text{otherwise}. \end{cases} \\ x_k^{(N)} &= = [\ldots\ 0\ \boxed{x_k}\ \underbrace{0\ 0\ \ldots 0}_{N-1}\ x_{N+k}\ 0\ \ldots]^\top. \end{aligned}$$

Take the above upsampled components, delay each by k, and sum them up; we get x back:

$$x_n = \sum_{k=0}^{N-1} x_{k,n-k}^{(N)}.$$

As this is a linear system, we can find the output as

$$\begin{aligned} y_n &= \sum_{k\in\mathbb{Z}} h_{k,n} \sum_{i=0}^{N-1} x_{i,n-i}^{(N)} = \sum_{i=0}^{N-1} \sum_{k\in\mathbb{Z}} h_{k,n}\, x_{i,n-i}^{(N)} \overset{(a)}{=} \sum_{i=0}^{N-1} \sum_{(k-i)/N\in\mathbb{Z}} h_{k,n}\, x_{i,n-i}^{(N)} \\ &\overset{(b)}{=} \sum_{i=0}^{N-1} \sum_{(k-i)/N\in\mathbb{Z}} h_{in}\, x_{i,n-i}^{(N)} \overset{(c)}{=} \sum_{i=0}^{N-1} \sum_{k\in\mathbb{Z}} h_{k,n}\, x_{i,n-i}^{(N)}, \end{aligned}$$

where (a) follows from the excluded terms being zero; (b) from the periodicity of h; and (c) because the added terms are zero. The final expression shows the output as a sum of N terms. The ith term is the ith polyphase component, upsampled and filtered by an i-sample delayed version of $h_{k,n}$.

3.26. *Sequence with a zero-polyphase component*
A sequence with all odd-indexed samples x equal to 0 is the upsampled-by-2 version of its first polyphase component, x_0. Using (3.92), its DTFT is given by

$$X(e^{j\omega}) = X_0(e^{j2\omega}),$$

so X is π-periodic since X_0 is 2π-periodic. Thus, if $X(e^{j\omega})$ is nonzero at $\omega = 0$, it is nonzero at $\omega = \pi$ as well.

3.27. *Convolution and sum of discrete random variables*
Since x and y are integer-valued, for any integer k, the event $\{\mathrm{z} = k\}$ is

$$\bigcup_{m\in\mathbb{Z}} \{\mathrm{x} = m,\ \mathrm{y} = k-m\}$$

as a union of disjoint events. Thus,

$$\begin{aligned} p_{\mathrm{z}}(k) &= \sum_{m\in\mathbb{Z}} \mathrm{P}(\mathrm{x} = m,\ \mathrm{y} = k-m) \overset{(a)}{=} \sum_{m\in\mathbb{Z}} \mathrm{P}(\mathrm{x} = m)\,\mathrm{P}(\mathrm{y} = k-m) \\ &\overset{(b)}{=} \sum_{m\in\mathbb{Z}} p_{\mathrm{x}}(m)\, p_{\mathrm{y}}(k-m) \overset{(c)}{=} (p_{\mathrm{x}} * p_{\mathrm{y}})(k), \end{aligned}$$

where (a) follows from the independence of x and y; (b) from the definition of PMF; and (c) from the definition of convolution.

3.28. *Autocorrelation and crosscorrelation*
We have

$$\begin{aligned} C_{\mathrm{x,y}}(e^{j\omega}) &= \sum_{k\in\mathbb{Z}} \mathrm{E}\big[\,\mathrm{x}_n\,\mathrm{y}^*_{n-k}\,\big]\, e^{-j\omega k} \;=\; \sum_{k\in\mathbb{Z}} \mathrm{E}\big[\,\mathrm{x}_n\,(\mathrm{x}_{n-k}+\mathrm{w}_{n-k})^*\,\big]\, e^{-j\omega k} \\ &= \sum_{k\in\mathbb{Z}} \big(\mathrm{E}\big[\,\mathrm{x}_n\,\mathrm{x}^*_{n-k}\,\big] + \mathrm{E}\big[\,\mathrm{x}_n\,\mathrm{w}^*_{n-k}\,\big]\big)\, e^{-j\omega k} \;\overset{(a)}{=}\; A_{\mathrm{x}}(e^{j\omega}), \end{aligned}$$

where (a) follows from x and w being uncorrelated. Also,

$$\begin{aligned} A_{\mathrm{y}}(e^{j\omega}) &= \sum_{k\in\mathbb{Z}} \mathrm{E}\big[\,\mathrm{y}_n\,\mathrm{y}^*_{n-k}\,\big]\, e^{-j\omega k} \;=\; \sum_{k\in\mathbb{Z}} \mathrm{E}\big[\,(\mathrm{x}_n+\mathrm{w}_n)\,(\mathrm{x}_{n-k}+\mathrm{w}_{n-k})^*\,\big]\, e^{-j\omega k} \\ &= \sum_{k\in\mathbb{Z}} \big(\mathrm{E}\big[\,\mathrm{x}_n\,\mathrm{x}^*_{n-k}\,\big] + \mathrm{E}\big[\,\mathrm{w}_n\,\mathrm{w}^*_{n-k}\,\big] + \mathrm{E}\big[\,\mathrm{x}_n\,\mathrm{w}^*_{n-k}\,\big] + \mathrm{E}\big[\,\mathrm{w}_n\,\mathrm{x}^*_{n-k}\,\big]\big)\, e^{-j\omega k} \\ &\overset{(a)}{=} A_{\mathrm{x}}(e^{j\omega}) + A_{\mathrm{w}}(e^{j\omega}), \end{aligned}$$

where (a) again follows from x and w being uncorrelated.

3.29. *Toeplitz matrix–vector products*
A size-$(N \times N)$ Toeplitz matrix has $1 + 2(N-1) = 2N-1$ parameters, while a same-size circulant matrix has N parameters. Thus, the minimal extension of a size-$(N\times N)$ Toeplitz matrix to a circulant matrix needs $N-1$ columns and rows.

The corresponding numbers of parameters for symmetric Toeplitz and circulant matrices are N and $\lfloor N+2/2 \rfloor$, respectively. So the minimal extension is by $N-2$ columns and rows.

For example, a Toeplitz matrix T extended to a circulant matrix C is as follows:

$$T \;=\; \begin{bmatrix} a & b & c \\ d & a & b \\ e & d & a \end{bmatrix} \qquad C \;=\; \begin{bmatrix} a & b & c & e & d \\ d & a & b & c & e \\ e & d & a & b & c \\ c & e & d & a & b \\ b & c & e & d & a \end{bmatrix},$$

while a symmetric Toeplitz matrix T extended to a symmetric circulant matrix C is as follows:

$$T \;=\; \begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix} \qquad C \;=\; \begin{bmatrix} a & b & c & b \\ b & a & b & c \\ c & b & a & b \\ b & c & b & a \end{bmatrix}.$$

Because we know that the DFT diagonalizes a circulant matrix, we can use that fact to estimate the cost of computing the product of a circulant matrix with a vector, and thus, the cost of computing the product of a Toeplitz matrix with a vector. Since the cost of computing the DFT is given by (3.271), the cost of computing the product of a Toeplitz matrix with a vector is $O(N \log_2 N)$.

3.30. *Overlap–save convolution algorithm*
In the factored form, three matrices are used, A, E and H as in Example 3.46,

$$A^\top \;=\; \begin{bmatrix} \ddots & & & & & & \\ & I_2 & 0 & 0 & & & \\ & 0 & I_2 & 0 & & & \\ & 0 & 0 & I_2 & & & \\ & & & I_2 & 0 & 0 & \\ & & & 0 & I_2 & 0 & \\ & & & 0 & 0 & I_2 & \\ & & & & & & \ddots \end{bmatrix}, \qquad E^\top \;=\; \begin{bmatrix} \ddots & & & & & & \\ & 0 & I_2 & 0 & & & \\ & 0 & 0 & I_2 & & & \\ & & & & 0 & I_2 & 0 \\ & & & & 0 & 0 & I_2 \\ & & & & & & & \ddots \end{bmatrix},$$

and H is given in (3.274). Multiplication by $A^\top$ will concatenate the following input blocks:

$$x^{(0)} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad x^{(1)} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}.$$

After multiplication by H_6, the output blocks will be

$$H_6\, x^{(0)} = \begin{bmatrix} h_0x_0 + h_2x_4 + h_1x_5 \\ h_1x_0 + h_0x_1 + h_2x_5 \\ h_2x_0 + h_1x_1 + h_0x_2 \\ h_2x_1 + h_1x_2 + h_0x_3 \\ h_2x_2 + h_1x_3 + h_0x_4 \\ h_2x_3 + h_1x_4 + h_0x_5 \end{bmatrix}, \qquad H_6\, x^{(1)} = \begin{bmatrix} h_0x_4 + h_2x_8 + h_1x_9 \\ h_1x_4 + h_0x_5 + h_2x_9 \\ h_2x_4 + h_1x_5 + h_0x_6 \\ h_2x_5 + h_1x_6 + h_0x_7 \\ h_2x_6 + h_1x_7 + h_0x_8 \\ h_2x_7 + h_1x_8 + h_0x_9 \end{bmatrix}.$$

We see that the first two elements of each block are incorrect while the last 4 are correct. Thus, the effect of $E^\top$ is to discard those incorrect elements. This is why this method is sometimes also called *overlap-discard*. In general, we keep the last M elements of each block.

3.31. *Sums and products*

(i)

$$\prod_{k=1}^{\infty} \exp\left(\frac{j2\pi}{k(k+1)}\right) = \exp\left(j2\pi \sum_{k=1}^{\infty} \frac{1}{k(k+1)}\right) \overset{(a)}{=} \exp(j2\pi) = 1,$$

where (a) follows from

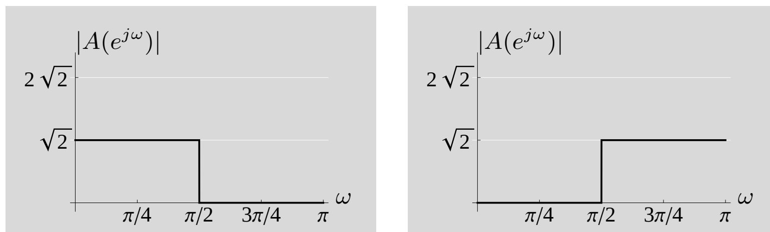
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

(ii)

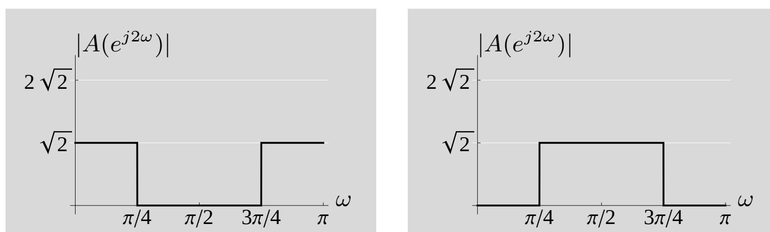
$$\sum_{k=0}^{1023} W_{16}^{k} = \sum_{k=0}^{63} \sum_{n=0}^{15} W_{16}^{16k+n} = \sum_{k=0}^{63} W_{16}^{16k} \sum_{n=0}^{15} W_{16}^{n} = 64 \sum_{n=0}^{15} W_{16}^{n} \overset{(a)}{=} 0,$$

where (a) follows from the orthogonality of the the roots of unity (3.288c).

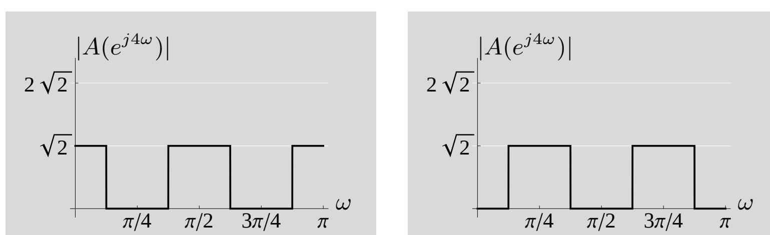
Ideal lowpass filter. Ideal highpass filter.



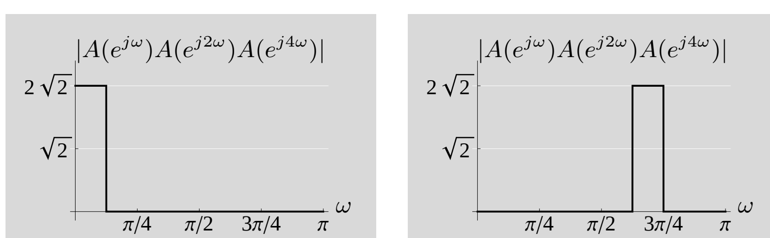
Magnitude responses.



Upsampled by 2.



Upsampled by 4.



Equivalent filters.

Figure S3.21-1 Iterated ideal filters.