How To Prove It: A Structured Approach
Third Edition
Solutions Manual

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Contents

Introduction

1. We use the formula $2^{ab} - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b})$ from the proof of Conjecture 2.
   (a) $2^{15} - 1 = 2^{3\cdot 5} - 1 = (2^5 - 1) \cdot (1 + 2^5 + 2^{10}) = 31 \cdot 1057$.
   (b) $2^{32,767} - 1 = 2^{1057\cdot 31} - 1 = (2^{31} - 1)(1 + 2^{31} + \cdots + 2^{1056\cdot 31})$. The first factor is $2^{31} - 1 = 2{,}147{,}483{,}647$.
2. When $n = 1$, $3^n - 1 = 2$, which is prime. For all $n > 1$, $3^n - 1$ is an even integer larger than 2, so it is not prime. If n is not prime, then $3^n - 2^n$ is not prime. If n is prime, then $3^n - 2^n$ may be either prime or composite.
3. (a) The method gives $m = 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$, which is prime.
   (b) The method gives $m = 2 \cdot 5 \cdot 11 + 1 = 111 = 3 \cdot 37$; 3 and 37 are both prime.
4. Using the method of the proof of Theorem 4, with $n = 5$, we get $x = 6! + 2 = 722$. The five consecutive composite numbers are $722 = 2 \cdot 361$, $723 = 3 \cdot 241$, $724 = 2 \cdot 362$, $725 = 5 \cdot 145$, and $726 = 2 \cdot 363$.
5. $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are Mersenne primes. Therefore, by Euclid's theorem, $2^4(2^5 - 1) = 496$ and $2^6(2^7 - 1) = 8128$ are both perfect.
6. No. The remainder when any integer $n > 3$ is divided by 3 will be either 0, 1, or 2. If it is 0, then n is divisible by 3, so it is composite. If it is 1, then $n + 2$ is divisible by 3 and therefore composite. If it is 2, then $n + 4$ is divisible by 3 and composite. Thus, the numbers n, $n + 2$, and $n + 4$ cannot all be prime.
7. The positive integers smaller than 220 that divide 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$. The positive integers smaller than 284 that divide 284 are 1, 2, 4, 71, and 142, and $1 + 2 + 4 + 71 + 142 = 220$.

Chapter 1

Section 1.1

1. (a) $(R \lor H) \land \neg(H \land T)$, where R stands for the statement "We'll have a reading assignment," H stands for "We'll have homework problems," and T stands for "We'll have a test."
   (b) $\neg G \lor (G \land \neg S)$, where G stands for "You'll go skiing," and S stands for "There will be snow."
   (c) $\neg[(\sqrt{7} < 2) \lor (\sqrt{7} = 2)]$.
2. (a) $(J \land B) \lor \neg(J \lor B)$, where J stands for "John is telling the truth" and B stands for "Bill is telling the truth."
   (b) $(F \lor C) \land \neg(F \land P)$, where F stands for "I'll have fish," C stands for "I'll have chicken," and P stands for "I'll have mashed potatoes."
   (c) $S \land N \land F$, where S stands for "6 is divisible by 3," N stands for "9 is divisible by 3," and F stands for "15 is divisible by 3."
3. Let A stand for the statement "Alice is in the room" and B for "Bob is in the room."
   (a) $\neg(A \land B)$.
   (b) $\neg A \land \neg B$.
   (c) $\neg A \lor \neg B$; this is equivalent to the formula in (a).
   (d) $\neg(A \lor B)$; this is equivalent to the formula in (b).
4. Let P stand for "Ralph is tall," Q for "Ed is tall," R for "Ralph is handsome," and S for "Ed is handsome."
   (a) $(P \land Q) \lor (R \land S)$.

(b) $(P \lor R) \land (Q \lor S)$.
(c) $\neg(P \lor R) \land \neg(Q \lor S)$.
(d) $\neg[(P \land R) \lor (Q \land S)]$.

5. (a) and (c) are well-formed formulas.

6. (a) I won't buy the pants without the shirt.
   (b) I won't buy the pants and I won't buy the shirt.
   (c) Either I won't buy the pants or I won't buy the shirt.

7. (a) Either Steve or George is happy, and either Steve or George is not happy.
   (b) Either Steve is happy, or George is happy and Steve isn't, or George isn't happy.
   (c) Either Steve is happy, or George is happy and either Steve or George isn't happy.

8. (a) Either taxes will go up or the deficit will go up.
   (b) Taxes and the deficit will not both go up, but it is not the case that taxes won't go up and the deficit also won't go up.
   (c) Either taxes will go up and the deficit won't, or the deficit will go up and taxes won't.

9. See exercise 7 in Section 1.2 for the determination of whether these arguments are valid.
   (a) Let J stand for "Jane will win the math prize," P for "Pete will win the math prize," and C for "Pete will win the chemistry prize." The premises are: $\neg(J \land P)$, $P \lor C$, J. The conclusion is C.
   (b) Let B stand for "The main course will be beef," F for "The main course will be fish," P for "The vegetable will be peas," and C for "The vegetable will be corn." The premises are: $B \lor F$, $P \lor C$, $\neg(F \land C)$. The conclusion is $\neg(B \land P)$.
   (c) Let J stand for "John is telling the truth," B for "Bill is telling the truth," and S for "Sam is telling the truth." The premises are $J \lor B$, $\neg S \lor \neg B$. The conclusion is $J \lor \neg S$.
   (d) Let S stand for "Sales will go up," E for "Expenses will go up," and B for "The boss will be happy." There is only one premise: $(S \land B) \lor (E \land \neg B)$. The conclusion is $\neg(S \land E)$.

Section 1.2

1. (a)

P	Q	$\neg P \lor Q$
F	F	T
F	T	T
T	F	F
T	T	T

   (b)

S	G	$(S \lor G) \land (\neg S \lor \neg G)$
F	F	F
F	T	T
T	F	T
T	T	F

2. (a)

P	Q	$\neg[P \land (Q \lor \neg P)]$
F	F	T
F	T	T
T	F	T
T	T	F

(b)

P	Q	R	$(P \vee Q) \wedge (\neg P \vee R)$
F	F	F	F
F	F	T	F
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

3. (a)

P	Q	$P + Q$
F	F	F
F	T	T
T	F	T
T	T	F

(b) Here are two possibilities: $(P \wedge \neg Q) \vee (Q \wedge \neg P)$, or $(P \vee Q) \wedge \neg(P \wedge Q)$. The truth table is the same as in part (a).

4. $\neg(\neg P \wedge \neg Q)$. The truth table is the same as the truth table for $P \vee Q$.

5. (a)

P	Q	$P \downarrow Q$
F	F	T
F	T	F
T	F	F
T	T	F

(b) $\neg(P \vee Q)$.

(c) $\neg P$ is equivalent to $P \downarrow P$, $P \vee Q$ is equivalent to $(P \downarrow Q) \downarrow (P \downarrow Q)$, and $P \wedge Q$ is equivalent to $(P \downarrow P) \downarrow (Q \downarrow Q)$.

6. (a)

P	Q	$P \mid Q$
F	F	T
F	T	T
T	F	T
T	T	F

(b) $\neg(P \wedge Q)$.

(c) $\neg P$ is equivalent to $P \mid P$, $P \vee Q$ is equivalent to $(P \mid P) \mid (Q \mid Q)$, and $P \wedge Q$ is equivalent to $(P \mid Q) \mid (P \mid Q)$.

7. We use the premises and conclusions identified in exercise 9 of Section 1.1.

(a) Valid: premises are all true only in line 6 of the following truth table, and the conclusion is also true in that line.

				Premises		Conclusion
J	P	C	$\neg(J \wedge P)$	$(P \vee C)$	J	C
F	F	F	T	F	F	F
F	F	T	T	T	F	T
F	T	F	T	T	F	F
F	T	T	T	T	F	T
T	F	F	T	F	T	F
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	T

(b) Invalid. Here is a line of the truth table in which all premises are true but the conclusion is false:

B	F	P	C	$B \vee F$ (Premises)	$P \vee C$ (Premises)	$\neg(F \wedge C)$ (Premises)	$\neg(B \wedge P)$ (Conclusion)
T	T	T	F	T	T	T	F

(c) Valid: premises are all true in lines 3, 5, 6, and 7 of the following truth table, and the conclusion is also true in all of those lines.

J	B	S	$J \vee B$ (Premises)	$\neg S \vee \neg B$ (Premises)	$J \vee \neg S$ (Conclusion)
F	F	F	F	T	T
F	F	T	F	T	F
F	T	F	T	T	T
F	T	T	T	F	F
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	F	T

(d) Invalid. Here is a line of the truth table in which the premises are all true but the conclusion is false:

S	E	B	$(S \wedge B) \vee (E \wedge \neg B)$ (Premise)	$\neg(S \wedge E)$ (Conclusion)
T	T	T	T	F

8. The following truth table shows that (a) and (c) are equivalent, as are (b) and (e):

P	Q	(a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$	(b) $\neg P \vee Q$	(c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$	(d) $\neg(P \vee Q)$	(e) $(Q \wedge P) \vee \neg P$
F	F	T	T	T	T	T
F	T	F	T	F	F	T
T	F	F	F	F	F	F
T	T	T	T	T	F	T

9. The following truth table shows that (b) is a contradiction and (c) is a tautology:

P	Q	(a) $(P \vee Q) \wedge (\neg P \vee \neg Q)$	(b) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$	(c) $(P \vee Q) \vee (\neg P \vee \neg Q)$
F	F	F	F	T
F	T	T	F	T
T	F	T	F	T
T	T	F	F	T

Formula (d) is also a tautology, as the following truth table shows:

P	Q	R	$[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	T

10. (a)

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	F	F

(b)

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	F	F	F	F
F	T	T	F	F	T	T
T	F	F	F	F	T	T
T	F	T	T	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

11. (a) $\neg(\neg P \wedge \neg Q)$ is equivalent to $\neg\neg P \vee \neg\neg Q$ (De Morgan's law),
which is equivalent to $P \vee Q$ (double negation law).

(b) $(P \wedge Q) \vee (P \wedge \neg Q)$ is equivalent to $P \wedge (Q \vee \neg Q)$ (distributive law),
which is equivalent to P (tautology law).

(c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$ is equivalent to $(\neg P \vee \neg\neg Q) \vee (\neg P \wedge Q)$ (De Morgan's law),
which is equivalent to $(\neg P \vee Q) \vee (\neg P \wedge Q)$ (double negation law),
which is equivalent to $\neg P \vee (Q \vee (\neg P \wedge Q))$ (associative law),
which is equivalent to $\neg P \vee (Q \vee (Q \wedge \neg P))$ (commutative law),
which is equivalent to $\neg P \vee Q$ (absorption law).

12. (a) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$
is equivalent to $(\neg\neg P \wedge \neg Q) \vee (P \wedge \neg R)$ (De Morgan's law),
which is equivalent to $(P \wedge \neg Q) \vee (P \wedge \neg R)$ (double negation law),
which is equivalent to $P \wedge (\neg Q \vee \neg R)$ (distributive law),
which is equivalent to $P \wedge \neg(Q \wedge R)$ (De Morgan's law).

(b) $\neg(\neg P \wedge Q) \vee (P \wedge \neg R)$
is equivalent to $(\neg\neg P \vee \neg Q) \vee (P \wedge \neg R)$ (De Morgan's law),
which is equivalent to $(P \vee \neg Q) \vee (P \wedge \neg R)$ (double negation law),
which is equivalent to $(\neg Q \vee P) \vee (P \wedge \neg R)$ (commutative law),
which is equivalent to $\neg Q \vee (P \vee (P \wedge \neg R))$ (associative law),
which is equivalent to $\neg Q \vee P$ (absorption law).

(c) $(P \wedge R) \vee [\neg R \wedge (P \vee Q)]$
is equivalent to $(P \wedge R) \vee [(\neg R \wedge P) \vee (\neg R \wedge Q)]$ (distributive law),
which is equivalent to $(P \wedge R) \vee [(P \wedge \neg R) \vee (Q \wedge \neg R)]$ (commutative law),
which is equivalent to $[(P \wedge R) \vee (P \wedge \neg R)] \vee (Q \wedge \neg R)$ (associative law),
which is equivalent to $[P \wedge (R \vee \neg R)] \vee (Q \wedge \neg R)$ (distributive law),
which is equivalent to $P \vee (Q \wedge \neg R)$ (tautology law).

13. $\neg(P \vee Q)$ is equivalent to $\neg(\neg\neg P \vee \neg\neg Q)$ (double negation law),
which is equivalent to $\neg\neg(\neg P \wedge \neg Q)$ (first De Morgan's law),
which is equivalent to $\neg P \wedge \neg Q$ (double negation law).

14. We use the associative law for $\wedge$ twice:

$$[P \wedge (Q \wedge R)] \wedge S \text{ is equivalent to } [(P \wedge Q) \wedge R] \wedge S$$
$$\text{which is equivalent to } (P \wedge Q) \wedge (R \wedge S).$$

15. 2^n.

16. $P \vee \neg Q$. (You can check this by making the truth table.)

17. $(P \wedge \neg Q) \vee (Q \wedge \neg P)$. (You can check this by making the truth table.)

18. If the conclusion is a tautology, then the argument is valid. If the conclusion is a contradiction and there is at least one line of the truth table in which all the premises are true, then the argument is not valid. If one of the premises is a tautology, then you can delete that premise without affecting the validity of the argument. If one of the premises is a contradiction, then the argument is valid.

Section 1.3

1. (a) $D(6) \wedge D(9) \wedge D(15)$, where $D(x)$ means "x is divisible by 3."
   (b) $D(x, 2) \wedge D(x, 3) \wedge \neg D(x, 4)$, where $D(x, y)$ means "x is divisible by y."
   (c) $N(x) \wedge N(y) \wedge [(P(x) \wedge \neg P(y)) \vee (P(y) \wedge \neg P(x))]$, where $N(x)$ means "x is a natural number" and $P(x)$ means "x is prime."
2. (a) $M(x) \wedge M(y) \wedge [(T(x, y) \vee T(y, x)]$, where $M(x)$ means "x is a man" and $T(x, y)$, means "x is taller than y."
   (b) $(B(x) \vee B(y)) \wedge (R(x) \vee R(y))$, where $B(x)$ means "x has brown eyes" and $R(x)$ means "x has red hair."
   (c) $(B(x) \wedge R(x)) \vee (B(y) \wedge R(y))$, where the letters have the same meanings as in part (b).
3. (a) $\{x \mid x \text{ is a planet}\}$.
   (b) $\{x \mid x \text{ is an Ivy League school}\}$.
   (c) $\{x \mid x \text{ is a state in the United States}\}$.
   (d) $\{x \mid x \text{ is a province or territory in Canada}\}$.
4. (a) $\{x \mid x \text{ is the square of a positive integer}\}$.
   (b) $\{x \mid x \text{ is a power of } 2\}$.
   (c) $\{x \mid x \text{ is an integer and } 10 \leq x < 20\}$.
5. (a) $(-3 \in \mathbb{R}) \wedge (13 - 2(-3) > 1)$. Bound variables: x; no free variables. This statement is true.
   (b) $(4 \in \mathbb{R}) \wedge (4 < 0) \wedge (13 - 2(4) > 1)$. Bound variables: x; no free variables. This statement is false.
   (c) $\neg[(5 \in \mathbb{R}) \wedge (13 - 2(5) > c)]$. Bound variables: x; free variables: c.
6. (a) $(w \in \mathbb{R}) \wedge (13 - 2w > c)$. Bound variables: x; free variables: w, c.
   (b) $(4 \in \mathbb{R}) \wedge (13 - 2(4) \text{ is a prime number})$. Bound variables: x, y; no free variables. This statement is true.
   (c) $(4 \text{ is a prime number}) \wedge (13 - 2(4) > 1)$. Bound variables: x, y; no free variables. This statement is false.
7. (a) $\{-1, 1/2\}$.

(b) $\{1/2\}$.

(c) $\{-1\}$.

(d) $\varnothing$.

8. (a) $\{x \mid$ Elizabeth Taylor was once married to $x\} = \{$Conrad Hilton Jr., Michael Wilding, Michael Todd, Eddie Fisher, Richard Burton, John Warner, Larry Fortensky$\}$.

(b) $\{x \mid x$ is a logical connective studied in Section 1.1$\} = \{\wedge, \vee, \neg\}$.

(c) $\{x \mid x$ is the author of this book$\} = \{$Daniel J. Velleman$\}$.

9. (a) $\{x \in \mathbb{R} \mid x^2 - 4x + 3 = 0\} = \{1, 3\}$.

(b) $\{x \in \mathbb{R} \mid x^2 - 2x + 3 = 0\} = \varnothing$.

(c) $\{x \in \mathbb{R} \mid 5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}\} = \{x \in \mathbb{R} \mid 5 \in \mathbb{R} \wedge x^2 + 25 < 50\} = \{x \in \mathbb{R} \mid x^2 < 25\} = \{x \in \mathbb{R} \mid -5 < x < 5\}$. Examples of elements: -4, 2, 4.9, π.

Section 1.4

1. (a) $\{3, 12\}$.

(b) $\{1, 12, 20, 35\}$.

(c) $\{1, 3, 12, 20, 35\}$.

The sets in parts (a) and (b) are both subsets of the set in part (c).

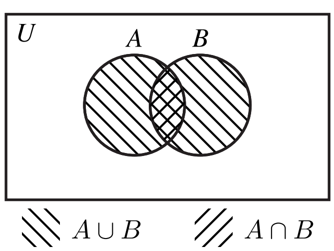
2. (a) $\{$United States, Germany, China, Australia, France, India, Brazil$\}$.

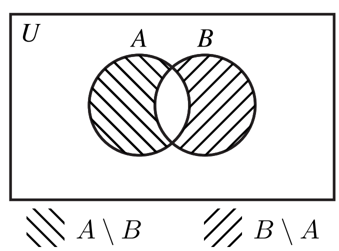
(b) $\varnothing$.

(c) $\{$France$\}$.

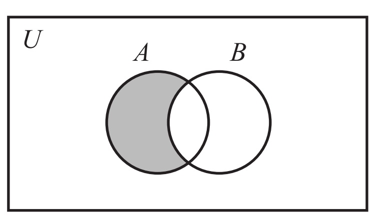
The set in part (b) is disjoint from both of the other sets, and also a subset of both other sets. The set in part (c) is a subset of the set in part (a).

3. In the diagram on the left below, the set $(A \cup B) \setminus (A \cap B)$ is represented by the region that is striped but not crosshatched. In the diagram on the right, the set $(A \setminus B) \cup (B \setminus A)$ is represented by the region that striped (in either direction).

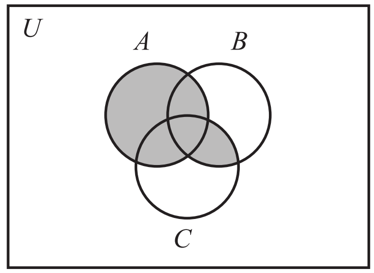




4. (a) Both Venn diagrams look like this:



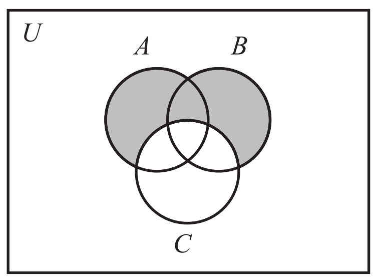
(b) Both Venn diagrams look like this:



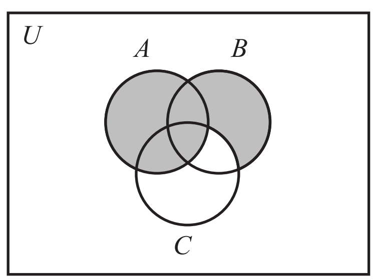
5. (a) $x \in A \setminus (A \cap B)$ is equivalent to $x \in A \wedge \neg(x \in A \wedge x \in B)$ (definitions of $\setminus$, $\cap$),
which is equivalent to $x \in A \wedge (x \notin A \vee x \notin B)$ (De Morgan's law),
which is equivalent to $(x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B)$ (distributive law),
which is equivalent to $x \in A \wedge x \notin B$ (contradiction law),
which is equivalent to $x \in A \setminus B$ (definition of $\setminus$).

(b) $x \in A \cup (B \cap C)$ is equivalent to $x \in A \vee (x \in B \wedge x \in C)$ (definitions of $\cup$, $\cap$),
which is equivalent to $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$ (distributive law),
which is equivalent to $x \in (A \cup B) \cap (A \cup C)$ (definitions of $\cup$, $\cap$).

6. (a) Both Venn diagrams look like this:



(b) Both Venn diagrams look like this:



7. (a) $x \in (A \cup B) \setminus C$ is equivalent to $(x \in A \vee x \in B) \wedge x \notin C$ (definitions of $\cup$, $\setminus$),
which is equivalent to $(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ (distributive law),
which is equivalent to $x \in (A \setminus C) \cup (B \setminus C)$ (definitions of $\setminus$, $\cup$).

(b) $x \in A \cup (B \setminus C)$ is equivalent to $x \in A \vee (x \in B \wedge x \notin C)$ (definitions of $\cup$, $\setminus$),
which is equivalent to $(x \in A \vee x \in B) \wedge (x \in A \vee x \notin C)$ (distributive law),
which is equivalent to $(x \in A \vee x \in B) \wedge \neg(x \notin A \wedge x \in C)$ (De Morgan's law),

which is equivalent to $x \in (A \cup B) \setminus (C \setminus A)$ (definitions of $\cup$, $\setminus$).

8. (a) $x \in (A \setminus B) \cap C$ is equivalent to $(x \in A \wedge x \notin B) \wedge x \in C$ (definitions of $\setminus$, $\cap$),
which is equivalent to $x \in A \wedge (x \notin B \wedge x \in C)$ (associative law),
which is equivalent to $x \in A \wedge (x \in C \wedge x \notin B)$ (commutative law),
which is equivalent to $(x \in A \wedge x \in C) \wedge x \notin B$ (associative law),
which is equivalent to $x \in (A \cap C) \setminus B$ (definitions of $\cap$, $\setminus$).

(b) $x \in (A \cap B) \setminus B$ is equivalent to $(x \in A \wedge x \in B) \wedge x \notin B$ (definitions of $\cap$, $\setminus$).
This is clearly a contradiction, so $(A \cap B) \setminus B = \varnothing$.

(c) $x \in A \setminus (A \setminus B)$ is equivalent to $x \in A \wedge \neg(x \in A \wedge x \notin B)$ (definition of $\setminus$),
which is equivalent to $x \in A \wedge (x \notin A \vee x \in B)$ (De Morgan's law),
which is equivalent to $(x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B)$ (distributive law),
which is equivalent to $x \in A \wedge x \in B$ (contradiction law),
which is equivalent to $x \in A \cap B$ (definition of $\cap$).

9. (a) $x \in (A \setminus B) \setminus C$ means $(x \in A \wedge x \notin B) \wedge x \notin C$.

(b) $x \in A \setminus (B \setminus C)$ means $x \in A \wedge \neg(x \in B \wedge x \notin C)$
which is equivalent to $x \in A \wedge (x \notin B \vee x \in C)$ (De Morgan's law),
which is equivalent to $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$ (distributive law).

(c) $x \in (A \setminus B) \cup (A \cap C)$ means $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$.

(d) $x \in (A \setminus B) \cap (A \setminus C)$ means $(x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$
which is equivalent to $((x \in A \wedge x \notin B) \wedge x \in A) \wedge x \notin C$ (associative law),
which is equivalent to $(x \in A \wedge (x \in A \wedge x \notin B)) \wedge x \notin C$ (commutative law),
which is equivalent to $((x \in A \wedge x \in A) \wedge x \notin B) \wedge x \notin C$ (associative law),
which is equivalent to $(x \in A \wedge x \notin B) \wedge x \notin C$ (idempotent law).

(e) $x \in A \setminus (B \cup C)$ means $x \in A \wedge \neg(x \in B \vee x \in C)$
which is equivalent to $x \in A \wedge (x \notin B \wedge x \notin C)$ (De Morgan's law),
which is equivalent to $(x \in A \wedge x \notin B) \wedge x \notin C$ (associative law).

This shows that sets (a), (d), and (e) are equal, and sets (b) and (c) are equal.

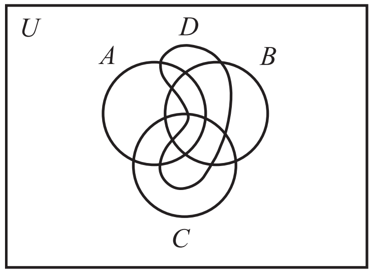
10. (a) $A = \{1, 2\}$, $B = \{2, 3\}$. $(A \cup B) \setminus B = \{1, 2, 3\} \setminus \{2, 3\} = \{1\} \neq \{1, 2\} = A$.

(b) $x \in (A \cup B) \setminus B$ is equivalent to $(x \in A \vee x \in B) \wedge x \notin B$ (definitions of $\cup$, $\setminus$),
which is equivalent to $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin B)$ (distributive law),
which is equivalent to $x \in A \wedge x \notin B$ (contradiction law),
which is equivalent to $x \in A \setminus B$ (definition of $\setminus$).

11. The Venn diagram for $(A \setminus B) \cup B$ looks the same as the Venn diagram for $A \cup B$, so $(A \setminus B) \cup B = A \cup B$. Therefore $A \subseteq (A \setminus B) \cup B$, but the two sets need not be equal. For example, if $A = \{1, 2\}$ and $B = \{2, 3\}$ then $(A \setminus B) \cup B = \{1\} \cup \{2, 3\} = \{1, 2, 3\} \neq \{1, 2\} = A$.

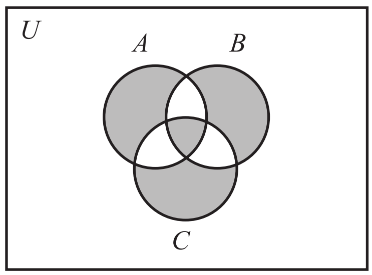
12. (a) There is no region corresponding to the set $(A \cap D) \setminus (B \cup C)$, but this set could have elements.

(b) Here is one possibility:



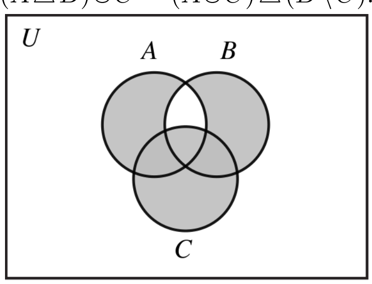
13. (a) The Venn diagrams are given in the solution to exercise 6. Those diagrams show that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.
    (b) $A = \{1,2,3\}$, $B = \{2,3,4\}$, $C = \{3,4,5\}$. $(A \cup B) \setminus C = \{1,2,3,4\} \setminus \{3,4,5\} = \{1,2\}$, $A \cup (B \setminus C) = \{1,2,3\} \cup \{2\} = \{1,2,3\}$.

14. The Venn diagrams for both sets look like this:

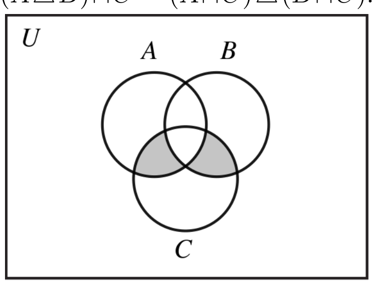


15\.

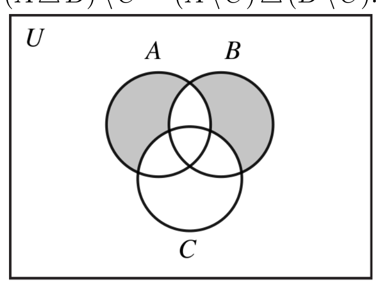
(a) $(A \triangle B) \cup C = (A \cup C) \triangle (B \setminus C)$.



(b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.

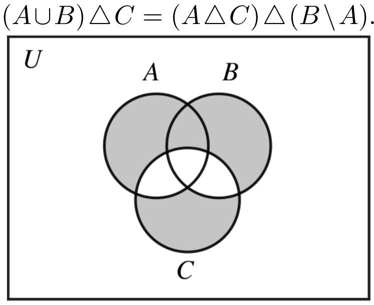


(c) $(A \triangle B) \setminus C = (A \setminus C) \triangle (B \setminus C)$.

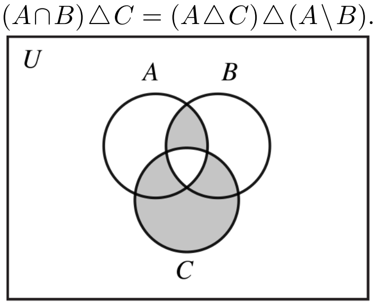


16\.

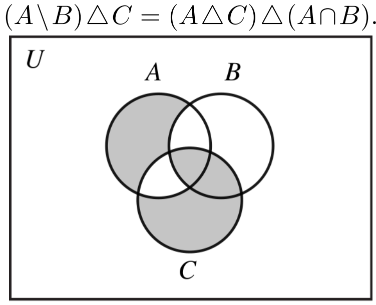
(a) $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$.



(b) $(A \cap B) \triangle C = (A \triangle C) \triangle (A \setminus B)$.



(c) $(A \setminus B) \triangle C = (A \triangle C) \triangle (A \cap B)$.



17. All of the following answers can be confirmed with Venn diagrams.
    (a) $C \setminus B$.
    (b) $C \setminus B$.
    (c) $A \cup B$.

Section 1.5

1. (a) $(S \vee \neg E) \rightarrow \neg H$, where S stands for "This gas has an unpleasant smell," E stands for "This gas is explosive," and H stands for "This gas is hydrogen."

   (b) $(F \wedge H) \rightarrow D$, where F stands for "George has a fever," H stands for "George has a headache," and D stands for "George will go to the doctor."

   (c) $(F \rightarrow D) \wedge (H \rightarrow D)$, where the letters have the same meanings as in part (b).

   (d) $(x \neq 2) \rightarrow (P(x) \rightarrow O(x))$, where $P(x)$ stands for "x is prime" and $O(x)$ stands for "x is odd."

2. (a) $H \rightarrow (P \wedge A)$, where H stands for "Mary will sell her house," P stands for "Mary can get a good price," and A stands for "Mary can find a nice apartment."

   (b) $M \rightarrow (C \wedge P)$, where C stands for "you have a good credit history," P stands for "you have an adequate down payment," and M stands for "you will get a mortgage."

   (c) $\neg S \rightarrow D$, where D stands for "John will drop out of school," and S stands for "someone will stop him."

   (d) $(D(x,4) \vee D(x,6)) \rightarrow \neg P(x)$, where $D(x,y)$ means "x is divisible by y" and $P(x)$ means "x is prime."

3. Let R stand for "It is raining," W for "It is windy," and S for "It is sunny."

   (a) $R \rightarrow (W \wedge \neg S)$.

   (b) $(W \wedge \neg S) \rightarrow R$. This is the converse of (a).

   (c) $R \rightarrow (W \wedge \neg S)$. This is the same as (a).

   (d) $(W \wedge \neg S) \rightarrow R$. This is the converse of (a).

   (e) $(S \vee \neg W) \rightarrow \neg R$

   which is equivalent to $R \rightarrow \neg(S \vee \neg W)$ (contrapositive law),
   which is equivalent to $R \rightarrow (\neg S \wedge W)$ (De Morgan's law),
   which is equivalent to (a).

   (f) $(R \rightarrow W) \wedge (R \rightarrow \neg S)$

   which is equivalent to $(\neg R \vee W) \wedge (\neg R \vee \neg S)$ (conditional law),
   which is equivalent to $\neg R \vee (W \wedge \neg S)$ (distributive law),
   which is equivalent to $R \rightarrow (W \wedge \neg S)$ (conditional law),
   which is the same as (a).

   (g) $(W \rightarrow R) \vee (\neg S \rightarrow R)$

   which is equivalent to $(\neg W \vee R) \vee (S \vee R)$ (conditional law),
   which is equivalent to $(\neg W \vee S) \vee (R \vee R)$ (commutative, associative laws),
   which is equivalent to $\neg(W \wedge \neg S) \vee R$ (De Morgan's, idempotent laws),
   which is equivalent to $(W \wedge \neg S) \rightarrow R$ (conditional law),
   which is the converse of (a).

4. (a) Let S stand for "Sales will go up," E for "Expenses will go up," and B for "The boss will be happy." Valid.

				Premises		Conclusion
S	E	B	$S \vee E$	$S \to B$	$E \to \neg B$	$\neg(S \wedge E)$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	T	T	T	T
F	T	T	T	T	F	T
T	F	F	T	F	T	T
T	F	T	T	T	T	T
T	T	F	T	F	T	F
T	T	T	T	T	F	F

(b) Let T stand for "The tax rate will go up," U for "The unemployment rate will go up," R for "There will be a recession," and G for "GDP will go up." Valid. To shorten the truth table, we take some shortcuts. "-" indicates an entry that could be T or F.

					Premises		Conclusion
T	U	R	G	$(T \wedge U) \to R$	$G \to \neg R$	$G \wedge T$	$\neg U$
F	-	-	-	T	-	F	-
T	-	-	F	-	T	F	-
T	F	-	T	T	-	T	T
T	T	F	T	F	T	T	F
T	T	T	T	T	F	T	F

(c) Let W stand for "The warning light will come on," P for "The pressure is too high," and C for "The relief valve is clogged." Invalid. Here is a line of the truth table in which the premises are all true but the conclusion is false:

			Premises		Conclusion
W	P	C	$W \leftrightarrow (P \wedge C)$	$\neg C$	$W \leftrightarrow P$
F	T	F	T	T	F

5. (a) Let J stand for "Jones will be convicted," P for "Jones will go to prison," and S for "Smith will testify agains Jones." Invalid. Here is a line of the truth table in which the premises are all true but the conclusion is false.

			Premises		Conclusion
J	P	S	$J \to P$	$J \to S$	$\neg S \to \neg P$
F	T	F	T	T	F

(b) Let D stand for "The Democrats will have a majority in the Senate," R for "The Republicans will have a majority in the senate," and B for "The bill will pass." Valid.

			Premises		Conclusion
D	R	B	$(D \vee R) \wedge \neg(D \wedge R)$	$B \to D$	$R \to \neg B$
F	F	F	F	T	T
F	F	T	F	F	T
F	T	F	T	T	T
F	T	T	T	F	F
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	F	T	T
T	T	T	F	T	F

6. (a)

P	Q	$P \leftrightarrow Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	T	T

(b) $(P \to Q) \lor (P \to R)$ is equivalent to $(\neg P \lor Q) \lor (\neg P \lor R)$ (conditional law),
which is equivalent to $(\neg P \lor \neg P) \lor (Q \lor R)$ (commutative, associative laws),
which is equivalent to $\neg P \lor (Q \lor R)$ (idempotent law),
which is equivalent to $P \to (Q \lor R)$ (conditional law).

7. (a) $(P \to R) \land (Q \to R)$
is equivalent to $(\neg P \lor R) \land (\neg Q \lor R)$ (conditional law),
which is equivalent to $(\neg P \land \neg Q) \lor R$ (distributive law),
which is equivalent to $\neg(P \lor Q) \lor R$ (De Morgan's law),
which is equivalent to $(P \lor Q) \to R$ (conditional law).

(b) $(P \to R) \lor (Q \to R)$
is equivalent to $(\neg P \lor R) \lor (\neg Q \lor R)$ (conditional law),
which is equivalent to $(\neg P \lor \neg Q) \lor (R \lor R)$ (commutative, associative laws),
which is equivalent to $\neg(P \land Q) \lor R$ (De Morgan's, idempotent laws),
which is equivalent to $(P \land Q) \to R$ (conditional law).

8. (a)

P	Q	R	$(P \to Q) \land (Q \to R)$	$(P \to R) \land [(P \leftrightarrow Q) \lor (R \leftrightarrow Q)]$
F	F	F	T	T
F	F	T	T	T
F	T	F	F	F
F	T	T	T	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T

(b) $(P \to Q) \lor (Q \to R)$
is equivalent to $(\neg P \lor Q) \lor (\neg Q \lor R)$ (conditional law),
which is equivalent to $(\neg P \lor R) \lor (Q \lor \neg Q)$ (commutative, associative laws),
which is a tautology (tautology law).

9. $\neg(P \to \neg Q)$. The equivalence can be checked with a truth table.

10. $\neg((P \to Q) \to \neg(Q \to P))$. The equivalence can be checked with a truth table.

11. (a)

P	Q	$(P \lor Q) \leftrightarrow Q$	$P \to Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

(b)

P	Q	$(P \land Q) \leftrightarrow Q$	$Q \to P$
F	F	T	T
F	T	F	F
T	F	T	T
T	T	T	T

12. (a), (b), and (d) are equivalent; (c) and (e) are equivalent. This can be seen in the following truth table.

P	Q	R	(a) $P \to (Q \to R)$	(b) $Q \to (P \to R)$	(c) $(P \to Q) \wedge (P \to R)$	(d) $(P \wedge Q) \to R$	(e) $P \to (Q \wedge R)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F
T	F	T	T	T	F	T	F
T	T	F	F	F	F	F	F
T	T	T	T	T	T	T	T

Chapter 2

Section 2.1

1. (a) $\forall x[\exists y F(x,y) \to S(x)]$, where $F(x,y)$ stands for "x has forgiven y," and $S(x)$ stands for "x is a saint."
   (b) $\neg\exists x[C(x) \wedge \forall y(D(y) \to S(x,y))]$, where $C(x)$ stands for "x is in the calculus class," $D(y)$ stands for "y is in the discrete math class," and $S(x,y)$ stands for "x is smarter than y."
   (c) $\forall x(\neg(x = m) \to L(x,m))$, where $L(x,y)$ stands for "x likes y," and m stands for Mary.
   (d) $\exists x(P(x) \wedge S(j,x)) \wedge \exists y(P(y) \wedge S(r,y))$, where $P(x)$ stands for "x is a police officer," $S(x,y)$ stands for "x saw y," j stands for Jane, and r stands for Roger.
   (e) $\exists x(P(x) \wedge S(j,x) \wedge S(r,x))$, where the letters have the same meanings as in part (d).
2. (a) $\forall x[B(x) \to \exists y(U(y,x) \wedge R(y))]$, where $B(x)$ stands for "x has bought a Rolls Royce with cash," $U(y,x)$ stands for "y is an uncle of x," and $R(y)$ stands for "y is rich."
   (b) $\exists x(D(x) \wedge M(x)) \to \forall x[\exists y(F(y,x) \wedge D(y)) \to Q(x)]$, where $D(x)$ stands for "x is in the dorm," $M(x)$ stands for "x has the measles," $F(y,x)$ stands for "y is a friend of x," and $Q(x)$ stands for "x will have to be quarantined."
   (c) $\neg\exists x F(x) \to \forall x[A(x) \to \exists y(D(y) \wedge T(x,y))]$, where $F(x)$ stands for "x failed the test," $A(x)$ stands for "x got an A," $D(x)$ stands for "x got a D," and $T(x,y)$ stands for "x will tutor y."
   (d) $\exists x D(x) \to D(j)$, where $D(x)$ stands for "x can do it" and j stands for Jones.
   (e) $D(j) \to \forall x D(x)$, where the letters have the same meanings as in part (d).
3. (a) $\forall z(z > x \to z > y)$. Free variables: x, y.
   (b) $\forall a[\exists x(ax^2 + 4x - 2 = 0) \leftrightarrow (a > -2 \vee a = -2)]$. No free variables.
   (c) $\forall x(x^3 - 3x < 3 \to x < 10)$. No free variables.
   (d) $(\exists x(x^2 + 5x = w) \wedge \exists y(4 - y^2 = w)) \to (-10 < w \wedge w < 10)$. Free variables: w.
4. (a) All unmarried men are unhappy.
   (b) y is a sister of one of x's parents; i.e., y is x's blood aunt.
5. (a) Every prime number other than 2 is odd.
   (b) There is a largest perfect number.
6. (a) There is no real number that is a solution to both of the equations $x^2+2x+3 = 0$ and $x^2+2x-3 = 0$. This is true, because $x^2 + 2x + 3$ and $x^2 + 2x - 3$ cannot both be 0.
   (b) It is not the case that both of the equations $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$ have real solutions. This is true, because the equation $x^2 + 2x + 3 = 0$ has no real solution.
   (c) Both of the equations $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$ have no real solutions. This is false, because $x = 1$ is a solution to the second equation.