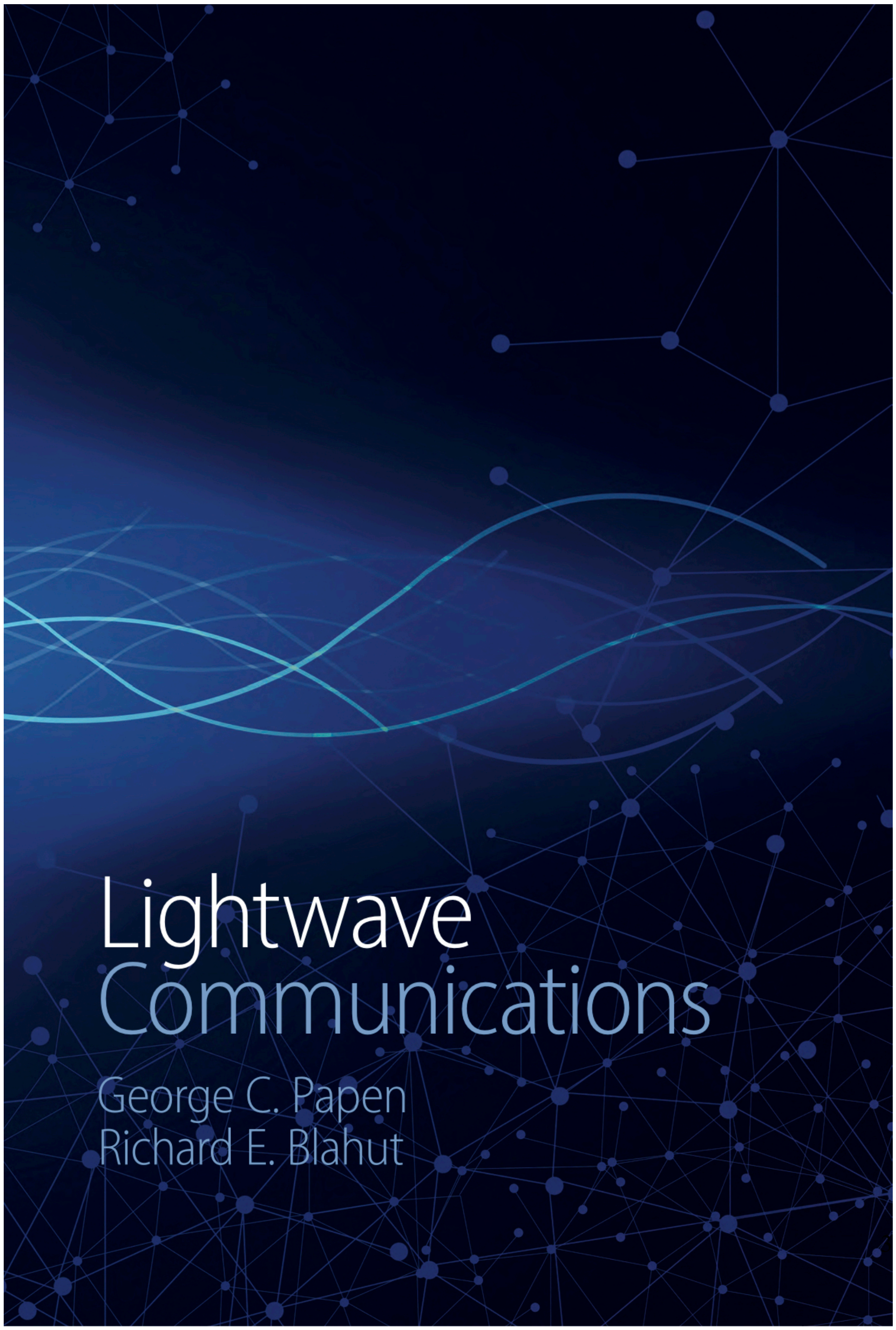
Selected Solutions for



Chapter 1 Selected Solutions

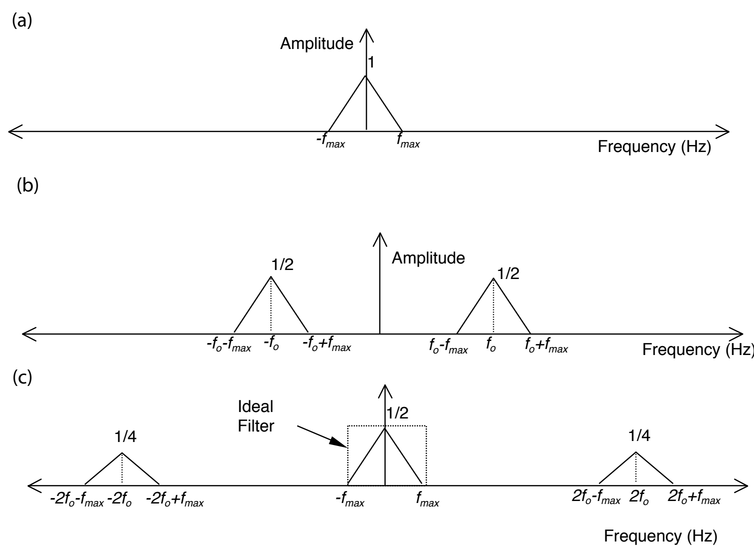
1.1 *Spectrum of an amplitude-modulated signal*

Let $s(t)$ be a bandlimited baseband signal with a frequency content $S(f)$ given by $1 - |f|/f_{\max}$ for $|f| < f_{\max}$, where $f_{\max}$ is the maximum frequency of the baseband signal. This baseband signal is multiplied by $\cos(2\pi f_c t)$ to produce an amplitude-modulated passband signal $\widetilde{s}(t) = s(t)\cos(2\pi f_c t)$, where f_c is the carrier frequency and f_c is much larger than $f_{\max}$.

(a) Sketch the frequency spectrum $S(f)$ of the baseband signal.

Solution

The baseband frequency spectrum is a triangular function shown in part (a) of the figure below.



(b) Sketch the frequency spectrum $\widetilde{S}(f)$ of the amplitude-modulated passband signal.

Solution

Applying the modulation property of Fourier transform (see Section 2.1), a sketch of the modulated spectrum is shown part (b) of the figure with $f_c = f_o$.

(c) The amplitude-modulated (AM) signal is demodulated by multiplying $\widetilde{s}(t)$ by a coherent signal of the form $\cos(2\pi f_c t)$. The signal is filtered by an ideal lowpass filter with a cutoff frequency $f_{\max}$. Sketch the magnitude of the frequency spectrum of the demodulated signal.

Solution

Applying the modulation property of Fourier transform again yields part (c) of the figure.

1.2 *Frequency demodulation errors*

This problem considers the effect of a frequency error in the process of demodulation. An amplitude-modulated signal $\tilde{s}(t) = s(t)\cos(2\pi f_c t)$ is demodulated using $\cos\big(2\pi f_c(1 + x)t\big)$, where x is a relative frequency error, and $s(t)$ is given in Problem 1. Sketch the magnitude of the frequency spectrum of the demodulated signal for: (a) $x = 0$, (b) $x = f_{\max}/10f_c$, and (c) $x = f_{\max}/f_c$. Comment on the results.

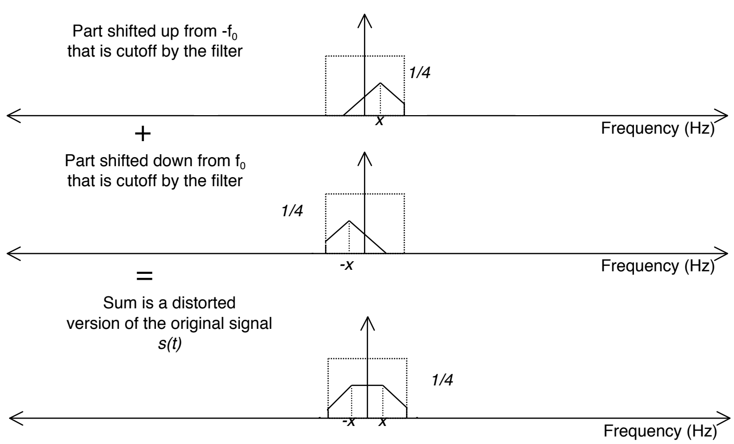
Solution

(a) This is the same as Problem 1c.

(b) and (c)

Solution

When $x \neq 0$, the two parts of the spectrum do not exactly overlap. This is shown schematically in the figure below with $f_c = f_o$.



When the offset x is equal to $f_{\max}$, the baseband spectrum is "inverted". The distortion is caused by the combination of the two parts of the spectrum not being aligned because of the different demodulation frequencies, and the lowpass filter which removes higher frequency components of the baseband signal.

1.3 *Phase demodulation errors*

An amplitude-modulated signal $\widetilde{s}(t) = A\cos(2\pi f_c t)$ is demodulated by using the reference $\cos(2\pi f_c t + \phi_e)$ where ϕ_e is a phase error, and A is a constant amplitude.

(a) Determine an expression for the demodulated signal as a function of the phase error ϕ_e.

Solution
Application of $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$ gives

$$\tfrac{1}{2}s(t)\cos\phi_e(t) + \tfrac{1}{2}s(t)\cos\bigl(4\pi f_0 t + \phi_e(t)\bigr).$$

When the phase variation is slow with respect to the carrier, the second term is filtered out by the lowpass filter. However, the first term shows that the original signal $s(t)$ is now multiplied by a phase error term $\cos\phi_e(t)$ producing distortion in the demodulated signal.

(b) The demodulated signal is now integrated over a time period T. What is the maximum phase error ϕ_e that can be tolerated for the demodulated signal to ensure that the magnitude is within 10% of the magnitude when there is no phase error?

Solution
To keep the magnitude of the error term within 10% requires $\cos\phi_e(t)$ to be less than 0.9. Solving for ϕ_e, gives $|\phi_e| < 0.45$ radians or $|\phi_e| < 25.8^o$.

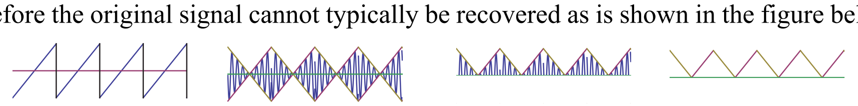
1.4 *Envelope demodulation*

Consider an amplitude-modulated passband signal

$$\widetilde{s}(t) = s(t)\cos(2\pi f_c t),$$

where $s(t) = \cos(2\pi f_1 t)$ and f_1 is much less than f_c. This signal is demodulated using envelope detection. What is the form of the resulting baseband signal in terms of $s(t)$?

Solution
This signal does not have a bias term and thus the envelope $s(t)$ is not always positive. Therefore the original signal cannot typically be recovered as is shown in the figure below.



Signal — Modulated signal with no bias — Rectified signal — Lowpass signal ("clipped" version of the original signal)

1.5 *Energy in the passband signal*

The energy in a real passband signal $\widetilde{s}(t)$ over an interval T is

$$E = \int_0^T \widetilde{s}(t)^2 \mathrm{d}t.$$

Let $\widetilde{s}(t) = A\cos(2\pi f_c t)$ and $T \gg 1/f_c$.

(a) Determine the energy in $\widetilde{s}(t)$ in terms of A and T.

Solution

$$\begin{aligned} E &= \int_0^T \widetilde{s}(t)^2 \mathrm{d}t \\ &= \int_0^T \left(A\cos(2\pi f_c t)\right)^2 \mathrm{d}t \\ &= \int_0^T \frac{A^2}{2}\left(1+\cos(4\pi f_c t)\right)\mathrm{d}t \\ &= \frac{A^2 T}{2}, \end{aligned}$$

where the second term on the third line is nearly zero when $T \gg 1/f_c$.

(b) Determine the energy in the demodulated baseband signal (cf.(1.3.3))

$$r(t) = \overline{A\cos(2\pi f_c t)\cdot\cos(2\pi f_c t)}$$

and compare this with the result in part (a).

Solution

When $T \gg 1/f_c$, $r(t) = \overline{A\cos(2\pi f_c t)\cdot\cos(2\pi f_c t)} = A/2$. Then

$$\begin{aligned} E &= \int_0^T r(t)^2 \mathrm{d}t \\ &= \int_0^T \frac{A^2}{4}\mathrm{d}t \\ &= \frac{A^2 T}{4}. \end{aligned}$$

The energy is half that of the incident signal.

(c) Compare both energies to that of a constant signal A over a time T.

Solution
The electrical energy is 1/4 that of a constant signal One factor of 1/2 is from modulation. The second factor of 1/2 is from demodulation.

1.6 *Sensitivity of a lightwave receiver*

Suppose that a phase-asynchronous lightwave communication system using a noncoherent carrier requires 10,000 photons per bit to achieve a bit error rate equal to 10^{-9}.

(a) If the information rate is 10 Gb/s, $\mathcal{R} = 1$, and λ = 1500 nm, what is the required lightwave signal power at the receiver? Express your answer in dBm.

Solution
Using the values given in the problem, the power is $\mathsf{m}hc_0/\lambda$ =13.3 μW which is -18.8 dBm.

(b) Determine the output photocurrent.

Solution
The current is given by $P\mathcal{R} = 13.3\,\mu$A.

(c) Determine the electrical power gain in decibels (dB) required after photodetection to produce a one volt signal into a 50 ohm resistor.

Solution
Generating a one volt signal into 50 Ω requires a current equal to $1/50$ A. Working with the current, the power gain of the electrical amplifier is $20\log_{10}((1/50)/13.3\times10^{-6}) = 64$ dB.

(d) A lightwave amplifier is now placed before the photodetector. Determine the lightwave signal power gain in decibels required to produce the same one-volt signal when no electrical amplification after photodetection is used.

Solution
Given the square-law nature of the photodetection process, the required gain for the optical amplifier is reduced by a factor of two when the gain is expressed in decibels. Therefore G is about 32 dB.

(e) Comment on the results of parts (c) and (d). (Note that there is a typo in letters denoting the problem sections the text.)

Solution

The square-law nature of the photodetection process means for the same photodetected current, the optical gain before direct photodetection is half the equivalent gain after direct photodetection in the electrical domain when the gain is expressed in decibels.

Chapter 2 Selected Solutions

2.1 *Linear systems*

Show that for any constants a and b, the definition of a linear system can be replaced by the single statement

$$a\,x_1(t) + b\,x_2(t) \to a\,y_1(t) + b\,y_2(t),$$

whenever $x_1(t) \to y_1(t)$, and $x_2(t) \to y_2(t)$.

Solution

Using homogeneity, input $a\,x_1(t)$ has output $b\,y_1(t)$. Let $a\,x_1(t) = X_1(t)$ and let $a\,y_1(t) = Y_1(t)$. Similar expressions can be derived for x_2 and y_2. Then using superposition gives

$$X_1(t) + X_2(t) \to Y_1(t) + Y_2(t).$$

Substituting back $X(t) = a\,x(t)$ and $Y(t) = b\,y(t)$ yields the desired expression.

2.2 *Properties of the Fourier transform*

(a) Starting with the definition of the Fourier transform and its inverse, derive the primary properties of the Fourier transform listed in Section 2.1.

Solution

<u>*Modulation Property*</u>

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}2\pi f t}e^{\mathrm{i}2\pi f_c t}\mathrm{d}t \\ = & \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}2\pi (f-f_c)t}\mathrm{d}t. \\ = & \ X(f-f_c) \end{aligned}$$

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<u>*Time Translation*</u>

Start with inverse transform

$$\begin{aligned} & \int_{-\infty}^{\infty} X(f)e^{\mathrm{i}2\pi f t}e^{-\mathrm{i}2\pi f t_0}\mathrm{d}f \\ = & \int_{-\infty}^{\infty} X(f)e^{\mathrm{i}2\pi (t-t_0)t}\mathrm{d}f \\ = & \ x(t-t_0). \end{aligned}$$

Scaling

$$\int_{-\infty}^{\infty} x(at)e^{-\mathrm{i}2\pi ft}\mathrm{d}t.$$

Let $t' = at$, then $\mathrm{d}t = \mathrm{d}t'/a$. Substituting gives

$$\begin{aligned} &= \frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-\mathrm{i}2\pi(f/a)t'}\mathrm{d}t' \\ &= \frac{1}{a}X(f/a). \end{aligned}$$

Differentiation

Write $x(t)$ as inverse transform and take the derivative of each side to give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} X(f)e^{\mathrm{i}2\pi ft}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} \mathrm{i}2\pi f X(f)e^{\mathrm{i}2\pi ft}\mathrm{d}f. \end{aligned}$$

(b) Using the modulation property of the Fourier transform and the transform pair $1 \longleftrightarrow \delta(f)$, show that $\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi f_1 t}e^{-\mathrm{i}2\pi f_2 t}\mathrm{d}t = \delta(f_2 - f_1)$, thereby demonstrating that the set $\{e^{-\mathrm{i}2\pi f_j t}\}$ of time-harmonic functions is orthogonal.

Solution

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi f_1 t}e^{-\mathrm{i}2\pi f_2 t}\mathrm{d}t &= \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi(f_1 - f_2)t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} (1)e^{-\mathrm{i}2\pi f' t}\mathrm{d}t \\ &= \delta(f') \\ &= \delta(f_1 - f_2), \end{aligned}$$

where $1 \longleftrightarrow \delta(f)$ has been used.

2.3 *Gram-Schmidt procedure*

The *Gram-Schmidt procedure* is a constructive method to create an orthonormal basis for the space spanned by a set of N signal vectors that are not necessarily linearly independent. Let $\{x_n(t)\}$ be a set of signal vectors. The procedure is as follows:

(a) Set $\psi_1(t) = x_1(t)/\sqrt{E_1}$ where E_1 is the signal energy.
(b) Determine the component of $x_2(t)$ that is linearly independent of $\psi_1(t)$ by finding the projection of $x_2(t)$ along $\psi_1(t)$. This component is given by $[x_2(t) \cdot \psi_1(t)]\psi_1(t)$ where the inner product is defined in (2.1.65).
(c) Subtract this component from $x_2(t)$.
(d) Normalize the difference. The resulting basis function can be written as

$$\psi_2(t) \quad = \quad \frac{x_2(t) - [x_2(t) \cdot \psi_1(t)]\,\psi_1(t)}{|x_2(t) - [x_2(t) \cdot \psi_1(t)]\,\psi_1(t)|}.$$

(e) Repeat for each subsequent function in the set forming the normalized difference between the function and the projection of the function onto each of the basis functions already determined. If the difference is zero, then the function is linearly dependent on the previous vectors and does not constitute a new basis vector.
(f) Continue until all functions have been used.

Using this procedure, determine:

(i) An orthonormal basis for the space over the interval $[0, 1]$ spanned by the functions $x_1(t) = 1$, $x_2(t) = \sin(2\pi t)$, and $x_3(t) = \cos^2(2\pi t)$.

Solution
The function $x_1(t) = 1$ is already normalized so $\psi_1(t) = x_1(t) = 1$. Project $x_2(t)$ onto $\psi_1(t)$ to give

$$\int_0^1 (1)\sin(2\pi t)\mathrm{d}t = 0,$$

showing that $x_2(t)$ is orthogonal to $\psi_1(t) = 1$. Normalizing this term gives

$$|x_2(t)| \quad = \quad \sqrt{\int_0^1 \sin^2(2\pi t)\mathrm{d}t} \;=\; \frac{1}{\sqrt{2}}$$

so that $\psi_2(t) = \sqrt{2}\sin(2\pi t)$. The last basis function is determined by expressing $\cos^2(2\pi t) = \frac{1}{2}(1 + \cos(4\pi t))$. Then

$$x_3(t) \cdot \psi_1(t) \quad = \quad \int_0^1 (1) \cdot \frac{1}{2}(1 + \cos(4\pi t))\,\mathrm{d}t \;=\; \frac{1}{2},$$

showing that the zero-frequency or DC component of $x_3(t)$ is the same as $\psi_1(t)$. Repeating for $\psi_3(t)$, we have

$$x_3(t) \cdot \psi_2(t) \;=\; \int_0^1 \sqrt{2} \sin(2\pi t) \cdot \frac{1}{2}(1+\cos(4\pi t))\,\mathrm{d}t \;=\; 0,$$

showing that $x_3(t)$ is orthogonal to $\psi_2(t)$. Therefore, the component of $x_3(t)$ that is orthogonal to both $\psi_1(t)$ and $\psi_2(t)$ is $\cos(4\pi t)$. Normalizing this term gives $\psi_3(t) = \sqrt{2}\cos(4\pi t)$.

(ii) An orthonormal basis for the space over the interval $[0, 1]$ spanned by the functions $x_1(t) = e^t$, $x_2(t) = e^{-t}$, and $x_3(t) = 1$.

Solution

Let $x_1(t) = e^t$ be the first function. Normalizing this term gives

$$|x_1(t)| \;=\; \sqrt{\int_0^1 e^{2t}\mathrm{d}t} \;=\; \sqrt{\tfrac{1}{2}(e^2-1)},$$

and thus

$$\psi_1(t) \;=\; \frac{e^t}{\sqrt{\frac{1}{2}(e^2-1)}}.$$

Now project this function onto the second function to give

$$x_2(t) \cdot \psi_1(t) \;=\; \frac{1}{\sqrt{\frac{1}{2}(e^2-1)}}\int_0^1 e^t e^{-t}\mathrm{d}t \;=\; \frac{1}{\sqrt{\frac{1}{2}(e^2-1)}}.$$

Subtracting from $x_2(t)$ gives

$$x_2(t) - (x_2(t) \cdot \psi_1(t))\,\psi_1(t) \;=\; e^{-t} - \frac{1}{\frac{1}{2}(e^2-1)}e^t.$$

Normalizing yields

$$\sqrt{\int_0^1 \left(e^{-t} - N_1 e_1^t\right)^2 \mathrm{d}t} \;=\; \sqrt{\frac{1}{2}\left(N_1^2(e^2-1) - 4N_1 - e^{-2} + 1\right.} \;=\; \frac{1}{N_2},$$

so that $\psi_2(t) = N_2(e^{-t} - N_1 e^t)$ where $N_1 = \frac{1}{\frac{1}{2}(e^2-1)}$ and N_2 are given above. Now project the third function $x_3(t) = 1$ onto each of first two functions to give

$$x_3(t) \cdot \psi_1(t) \;=\; N_1 \int_0^1 e^t \mathrm{d}t \;=\; N_1(e-1),$$

and

$$x_3(t) \cdot \psi_2(t) \quad = \quad N_2 \int_0^1 (e^{-t} - N_1 e^t)\mathrm{d}t \; = \; \frac{N_2(N_1 e + 1)(e-1)}{e}.$$

Subtract to produce the part of the third function that is orthogonal to both $\psi_1(t)$ and $\psi_2(t)$

$$\begin{aligned} & x_3(t) - \big([x_3(t) \cdot \psi_1(t)]\, \psi_1(t) + [x_3(t) \cdot \psi_2(t)]\, \psi_2(t)\big) \\ = \;& 1 - \left(N_1\,(e-1)\,N_1 e^{-t}(t) + \frac{N_2(N_1 e + 1)(e-1)}{e} N_2(e^{-t} - N_1 e^t)\right) \\ = \;& 1 - N_1^2\left(e^{(1-t)} - 1\right) + N_2^2(N_1 e + 1)(e-1)(e^{-(t+1)} - N_1 e^{t-1}) \end{aligned}$$

This function can then be normalized to determine $\psi_3(t)$.

2.4 *Gaussian pulse*

(a) Using the Fourier transform pair $e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$ and the scaling property of the Fourier transform, show that

$$e^{-t^2/2\sigma^2} \quad \longleftrightarrow \quad \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2} \quad = \quad \sqrt{2\pi}\sigma e^{-\sigma^2\omega^2/2}.$$

Solution
Let $c^{-1} = \sqrt{2\pi}\sigma$. Using the scaling property of Fourier transforms gives

$$e^{-\pi(ct)^2} \longleftrightarrow \frac{1}{c} e^{-\pi(f/c)^2}$$

so that

$$\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}.$$

Using $\omega = 2\pi f$, this expression shows the exact reciprocal relationship between the root-mean squared timewidth and the root-mean squared bandwidth expressed in (angular) frequency.

(b) Using an angular frequency ω, show that when the root-mean-squared timewidth is defined using the squared-magnitude of the root-mean-squared bandwidth is defined using the squared-magnitude of the Fourier transform, $T_{\text{rms}}\mathcal{W}_{\text{rms}} = 1/2$.

Solution
When the squared magnitude of the pulse is used, $T_{\text{rms}} = \sigma/\sqrt{2}$. Then, using an angular frequency ω, $\mathcal{W}_{\text{rms}} = 1/\sqrt{2}\sigma$. Therefore $T_{\text{rms}}\mathcal{W}_{\text{rms}} = 1/2$. This relationship is the basis

for the Heisenberg uncertainty relationship discussed in Chapter 15

(c) Derive the relationship between the root-mean-squared bandwidth $\mathcal{W}_{\text{rms}}$ for the signal power and the -3 dB or half-power bandwidth $\mathcal{W}_h$ for a pulse whose power $P(t)$ is given by $e^{-t^2/2\sigma_P^2}$.

Solution

The 3 dB point is defined when the frequency function is half the peak value at $f = 0$. Using the Fourier transform pair derived in part (a), and solving for f gives

$$\begin{aligned} e^{-2\pi^2\sigma_P^2 f^2} &= \frac{1}{2} \\ \Rightarrow \quad f_{\text{3dB}} &= \frac{1}{\sigma\pi}\sqrt{\frac{\log_e(2)}{2}}. \end{aligned}$$

(d) A lightwave pulse $s(t)$ modeled as a gaussian pulse with a root-mean-squared timewidth T_{rms} is incident on a square-law photodetector with the electrical pulse $p(t)$ generated by direct photodetection given by $|s(t)|^2/2$. Determine the following:

(i) The root-mean-squared timewidth of $p(t)$ in terms of T_{rms}.

Solution

Squaring an gaussian reduces the root-mean timewidth by a factor of $\sqrt{2}$. Therefore, the root-mean-squared timewidth of $p(t)$ is equal to $T_{\text{rms}}/\sqrt{2}$.

(ii) The root-mean-squared timewidth of the electrical power per unit resistance $P_e(t) = p(t)^2$ in terms of T_{rms}.

Solution

The signal is squared again so that the root-mean squared width of the electrical power pulse $P_e(t) = p^2(t)$ is half the root-mean squared width of the lightwave pulse.

(e) Finally, rank order the root-mean-squared timewidth of the lightwave pulse $s(t)$, the electrical pulse $p(t)$ generated by direct photodetection, and the electrical power pulse $P_e(t)$. Are these results valid for any kind of pulse?

Solution

The order from largest to smallest is: the lightwave pulse, the photodetected pulse, and then the electrical power pulse. The results are valid for any kind of smooth pulse such that the derivative of the pulse does not contain impulses.

2.5 *Pulse formats*

Derive relationships between the root-mean-squared width, the -3 dB width, and the full-width-half-maximum width in both the time domain and the frequency domain for:

(a) A *rectangular pulse* defined as $p(t) = 1$ for $-W/2 \le t \le W/2$, and zero otherwise.

Solution

For the rectangular pulse, the full width half max (FWHM) is equal to the width W of the pulse. The definition of the root-mean squared width is (cf. (2.1.30))

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t-\bar{t})^2 p(t)\mathrm{d}t}{\int_{-\infty}^{\infty} p(t)\mathrm{d}t} \qquad \text{where } \bar{t} = \frac{\int_{-\infty}^{\infty} t p(t)\mathrm{d}t}{\int_{-\infty}^{\infty} p(t)\mathrm{d}t}.$$

Because it is a square pulse of unit height and base W, the area is W. Because the function is even, $\bar{t} = 0$. Therefore,

$$\begin{aligned}
\sigma_t^2 &= \frac{\int_{-\infty}^{\infty} (t-\bar{t})^2 p(t)\mathrm{d}t}{\int_{-\infty}^{\infty} p(t)\mathrm{d}t} \\
&= \frac{\int_{-\infty}^{\infty} t^2 p(t)\mathrm{d}t}{W} \\
&= \frac{\int_{-\frac{W}{2}}^{\frac{W}{2}} t^2\mathrm{d}t}{W} = \frac{W^2}{12} \\
\rightarrow \sigma_t &= \frac{W}{\sqrt{12}}.
\end{aligned}$$

(b) A *triangular pulse* defined as $p(t) = 1 - |t|/W$ for $|t| \le W$, and zero otherwise.

Solution

For this case, the full width half max (FWHM) is half the base $2W$ of the triangle, or W. Because it is a triangular pulse of unit height and base $2W$, the area is W as before. Because the function is even, $\bar{t} = 0$ as before. Therefore,

$$\begin{aligned}
\sigma_t^2 &= \frac{\int_{-\infty}^{\infty}(t-\bar{t})^2 p(t)\mathrm{d}t}{\int_{-\infty}^{\infty}p(t)\mathrm{d}t} \\
&= \frac{\int_{-\infty}^{\infty}t^2 p(t)\mathrm{d}t}{W} \\
&= \frac{2\int_0^W t^2\big(1-\frac{t}{W}\big)\mathrm{d}t}{W} \quad = \quad \frac{W^2}{6} \\
&\to \frac{W}{\sqrt{6}}.
\end{aligned}$$

This width is a factor of $\sqrt{2}$ less than the width of the rectangular pulse. This factor can be explained by noting that a triangular pulse is the convolution of a square pulse with itself. (See Problem 2.6)

(b) A lorentzian pulse defined as

$$p(t) \quad = \quad \frac{2\alpha}{t^2+\alpha^2},$$

where α is a constant.

Solution

The full-width half-maximum value is defined when $p(t) = 1/2$. Solving gives t_{FWHM} as $\sqrt{\alpha(4-\alpha)}$. The root-mean-squared width of a lorentzian pulse is

$$T_{\text{rms}}^2 \quad = \quad \overline{t^2} \quad = \quad = \frac{1}{E}\int_{-\infty}^{\infty}\frac{2\alpha t^2}{t^2+\alpha^2}\mathrm{d}t,$$

where $E = \int_{-\infty}^{\infty}|p(t)|^2\mathrm{d}t$ is the pulse energy. Because the integrand goes to the constant 2α as t goes to infinity, the integral does not converge and the root-mean squared timewidth for a lorentzian pulse is not defined or is defined as infinite.

2.6 *Pulse characterization*

The rectangular pulse $p(t)$ defined in Problem 2.5 is used as the input to a time-invariant linear system defined by $h(t) = p(t)$ so that the impulse response is equal to the input pulse.

(a) Derive the full-width-half-maximum timewidth and the root-mean-squared timewidth of the output $y(t) = x(t) \circledast h(t)$ and show explicitly that $2\sigma_p^2 = \sigma_y^2$ where σ is the root-mean-squared timewidth.

Solution
The convolution of the input square pulse $h(t)$ with an impulse response $g(t)$ that has the same functional shape results in a triangular output pulse of the same form as Problem 2.5(b).

Therefore, using the root-mean squared width from part Problem 2.5(a), we have

$$h_{\text{rms}}^2 + g_{\text{rms}}^2 = \frac{W^2}{12} + \frac{W^2}{12} = \frac{W^2}{6},$$

which is the square of the root-mean squared width $g(t)$ from Problem 2.5(b), showing that the root-mean squared timewidths of optical pulses "add in quadrature" for nonnegative pulse shapes according to $h_{\text{rms}}^2 + g_{\text{rms}}^2 = f_{\text{rms}}^2$.

(b) Let the full-width-half-maximum width be denoted by F. Determine whether the relationship $2F_p^2 = F_y^2$ holds for each pulse defined in Problem 2.5.

Solution
Using the definition of the full-width-half-maximum width gives

$$h_{\text{FWHM}}^2 + g_{\text{FWHM}}^2 = W^2 + W^2 = 2W^2,$$

which does not equal the full-width-half-maximum width of the output pulse for any pulse considered in this problem. For this reason, timewidths and bandwidths based on root-mean squared values are often preferred to full-width-half-maximum widths.

2.7 *Passband, baseband, analytic signals, and the Hilbert transform*

(a) Using

$$\begin{aligned}\widetilde{s}(t) &= A(t)\cos\big(2\pi f_c t + \phi(t)\big) \\ &= \text{Re}\left[\left(s_I(t) + \mathrm{i}s_Q(t)\right)e^{\mathrm{i}2\pi f_c t}\right] \\ &= \text{Re}[z(t)],\end{aligned}$$

determine expressions for $A(t)$ and $\phi(t)$ in terms of $s_I(t)$ and $s_Q(t)$.

Solution
The relationship is just the conversion between rectangular and polar coordinates

$$A(t) = \sqrt{s_I(t)^2 + s_Q(t)^2}$$

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right].$$

(b) Verify the following relationships:

(i) $s_I(t) = \text{Re}\left[z(t)e^{-\text{i}2\pi f_c t}\right]$

Solution
When $\widetilde{s}(t) = A(t)\cos\left[2\pi f_c t + \phi(t)\right]$, then by definition $\widetilde{s}(t) = \text{Re}[z(t)]$, and $z(t) = A(t)e^{\text{i}(2\pi f_c t+\phi(t))}$. Therefore, $\text{Re}\left[A(t)e^{\text{i}(2\pi f_c t+\phi(t))}e^{-\text{i}2\pi f_c t}\right] = \text{Re}\left[A(t)e^{\text{i}\phi(t)}\right] = A(t)\cos\phi(t) = s_I(t)$.

(ii) $s_Q(t) = \text{Im}\left[z(t)e^{-\text{i}2\pi f_c t}\right]$

Solution
$\text{Im}\left[A(t)e^{\text{i}(2\pi f_c t+\phi(t))}e^{-\text{i}2\pi f_c t}\right] = \text{Im}\left[A(t)e^{\text{i}\phi(t)}\right] = A(t)\sin\phi(t) = s_Q(t)$.

(iii) $A(t) = |z(t)|$

Solution
Follows directly from the definition of $z(t)$.

(iv) $\phi(t) = \arg\left(z(t)e^{-\text{i}2\pi f_c t}\right)$

Solution
Follows directly from the definition of $z(t)$.

(c) Derive a relationship for the Hilbert transform $\widehat{s}(t)$ in terms of the complex-baseband signal $s_I(t) + \text{i}s_Q(t)$ and the carrier frequency f_c.

Solution
By definition $\widehat{x}(t) = \text{Im}[z(t)]$ (cf. (2.1.22)) so that $\widehat{x}(t) = \text{Im}[A(t)e^{\phi(t)}e^{\text{i}2\pi f_c t}]$ or
$\widehat{x}(t) = \text{Im}\left[\left(s_I(t) + \text{i}s_Q(t)\right)e^{\text{i}2\pi f_c t}\right]$

2.16 *Marginalization*

The bivariate gaussian probability density function has the form

$$p_{\underline{x},\underline{y}}(x,y) \quad = \quad Ae^{-(ax^2+2bxy+cy^2)}.$$

(a) Express A in terms of a, b, and c.

Solution

Express the joint probability density function in the standard form of a multivariant gaussian distribution

$$f_{\underline{x},\underline{y}}(x,y) = \frac{1}{\sqrt{(2\pi)^N \det \mathbb{C}}} e^{-\frac{1}{2}(\mathbf{x}-\langle\mathbf{x}\rangle)^T \mathbb{C}^{-1}(\mathbf{x}-\langle\mathbf{x}\rangle)},$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Comparing this standard form to the expression given in the problem, it is seen that the means $\langle x\rangle$ and $\langle y\rangle$ are both zero because there is no constant term. Moreover, the inverse of the covariance matrix $\mathbb{C}^{-1}$ is

$$\mathbb{C}^{-1} = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

with the covariance matrix given by

$$\mathbb{C} = \frac{1}{2(ac-b^2)}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix},$$

and the determinant of $\mathbb{C}$ given by

$$\det \mathbb{C} = \frac{1}{4(ac-b^2)}.$$

The variances are diagonal terms of the covariance matrix and are given by

$$\sigma_x^2 = \frac{c}{4(ac-b^2)} \qquad \sigma_y^2 = \frac{a}{4(ac-b^2)}.$$

Using these expressions, the normalization is

$$A = \frac{1}{\sqrt{(2\pi)^2 \det \mathbb{C}}} = \frac{\sqrt{ac-b^2}}{\pi}.$$

(b) Find the marginals, $p_{\underline{x}}(x)$ and $p_{\underline{y}}(y)$, and the conditionals $p_{\underline{x}|\underline{y}}(x|y)$ and $p_{\underline{y}|\underline{x}}(y|x)$. (c) Find the means $\langle\underline{x}\rangle$, $\langle\underline{y}\rangle$, the variances $\sigma_{\underline{x}}^2$, $\sigma_{\underline{y}}^2$, and the correlation $\langle\underline{x}\underline{y}\rangle$.

Solution

The means and the variances were derived in part (a). The marginals are given by

$$f_{\underline{x}}(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-x^2/2\sigma_x^2} \qquad f_{\underline{y}}(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2/2\sigma_y^2}$$

The correlation $\langle \underline{x}\underline{y} \rangle$ is

$$\langle \underline{x}\underline{y} \rangle = \frac{\sqrt{ac-b^2}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy e^{-(ax^2+2bxy+cy^2)} \mathrm{d}x\mathrm{d}y = -\frac{b}{2(ac-b^2)},$$

which is simply the off-diagonal element of the covariance matrix. Finally, the conditional distributions are

$$f_{\underline{x}|\underline{y}}(x|y) = \frac{f_{\underline{x},\underline{y}}(x,y)}{f_{\underline{y}}(y)} \qquad\qquad p_{\underline{y}|\underline{x}}(y|x) = \frac{f_{\underline{x},\underline{y}}(x,y)}{f_{\underline{x}}(x)}.$$

2.18 *Joint and marginal gaussian probability density functions*

The joint probability density function $p(x,y)$ is given as

$$p(x,y) = \begin{cases} \dfrac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\dfrac{1}{2}\left(\dfrac{x^2}{2\sigma_x^2} + \dfrac{y^2}{2\sigma_y^2}\right)\right] & \text{if } xy > 0 \\ 0 & \text{if } xy < 0 \end{cases} .$$

(a) Show that this function is a valid probability density function.

Solution

The integral of the joint probability distribution separates to an integral over x and an integral over y. The integral on either x or y is half the value over the whole plane. Therefore

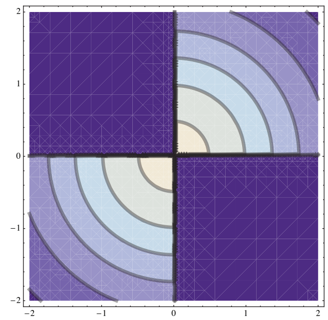
$$\begin{aligned} p_{\underline{x}}(x) &= \frac{1}{2}\int_{\infty}^{\infty} p_{\underline{x}\underline{y}}(x,y)dx \\ &= \frac{1}{4\pi\sigma_x\sigma_y} \exp\left[-\frac{y^2}{4\sigma_y^2}\right] \underbrace{\int_{\infty}^{\infty} \exp\left[-\frac{x^2}{4\sigma_x^2}\right] dx}_{2\sigma_x\sqrt{\pi}} \\ &= \frac{1}{2\sqrt{\pi}\sigma_y} \exp\left[-\frac{y^2}{4\sigma_y^2}\right]. \end{aligned}$$

Including an additional factor of two because of symmetry, this marginal probability density function integrates to one so that the distribution is a valid probability density function.

(b) Sketch $p(x,y)$ in plan view and in three dimensions. Is this joint probability density function jointly gaussian?

Solution

The plot of the function is on the next page for $\sigma_x = \sigma_y = 1$.



It is nonzero in the first and third quadrants when $xy > 0$. It is zero elsewhere.This joint probability density function is not jointly gaussian.

(c) Find the marginal probability density functions $p_{\underline{x}}(x)$ and $p_{\underline{y}}(y)$ and comment on this result.

Solution

The marginal distribution for y is of the same form as the form derived in part (a) with each marginal distribution equal to a gaussian distribution. Therefore, knowing that each marginal distribution is gaussian is not sufficient to infer that the joint distribution is jointly gaussian.

2.21 *Coherence function and the power density spectrum*

(a) Let $R(\tau) = e^{-|\tau|}e^{\mathrm{i}2\pi f_c \tau}$. Determine the one-sided power density spectra $\mathcal{S}(f)$ and $\mathcal{S}_\lambda(\lambda)$.

Solution

The power spectral density $S_f(f)$ is the Fourier transform of the coherence function $r(\tau)$

$$
\begin{aligned}
S_f(f) &= \int_{-\infty}^{\infty} r(\tau)e^{-\mathrm{i}2\pi f\tau}\mathrm{d}\tau \\
&= \int_{-\infty}^{0} e^{\tau}e^{\mathrm{i}2\pi f_c\tau}e^{-\mathrm{i}2\pi f\tau}\mathrm{d}\tau + \int_{0}^{\infty} e^{-\tau}e^{\mathrm{i}2\pi f_c\tau}e^{-\mathrm{i}2\pi f\tau}\mathrm{d}\tau \\
&= = \frac{\mathrm{i}}{2\pi(f-f_c)+\mathrm{i}} + \frac{\mathrm{i}}{2\pi(f_c-f)+\mathrm{i}} \\
&= \frac{2}{1+4\pi^2(f-f_c)^2},
\end{aligned}
$$

which is a shifted lorentzian spectrum. The one-sided spectrum is twice this value and is defined for nonnegative frequencies. To find $S_\lambda(\lambda)$, use $S_\lambda(\lambda)\,\mathrm{d}\lambda = S_f(f)\,\mathrm{d}f$ to obtain

$$\begin{aligned}
S_\lambda(\lambda) &= S_f(f)\frac{\mathrm{d}f}{d\lambda} \\
&= -\frac{2}{1+4\pi^2(f-f_c)^2}\frac{c}{\lambda^2} \\
&= -\frac{c}{\lambda^2}\frac{2}{1+4\pi^2 c\left(\frac{1}{\lambda}-\frac{1}{\lambda_c}\right)^2} \\
&= -\frac{2c}{\lambda^2+\frac{4\pi^2 c}{\lambda_c}(\lambda_c-\lambda)^2}.
\end{aligned}$$

(b) A lightwave carrier has a power density spectrum $\mathcal{S}_\lambda(\lambda)$ given by

$$S_\lambda(\lambda) = \frac{\pi}{(\lambda-\lambda_c)^2+\pi}.$$

Determine the total lightwave signal power P.

Solution

Let $x = \lambda - \lambda_c$ where λ_c is a constant. Then $\mathrm{d}\lambda = \mathrm{d}x$. The spectral density is one-sided so the limits on λ are 0 and ∞. When $\lambda = 0$, $x = -\lambda_c$. The upper limit remains the same so that the integral is

$$\int_0^\infty \frac{\pi}{(\lambda-\lambda_c)^2+\pi}\mathrm{d}\lambda = \pi\int_{-\lambda_c}^\infty \frac{1}{x^2+\pi^2}\mathrm{d}x.$$

The integral is the form of an arctan function so that

$$\begin{aligned}
\pi\int_{-\lambda_c}^\infty \frac{1}{x^2+\pi^2}\mathrm{d}x &= \sqrt{\pi}\left|\arctan\left(\frac{x}{\sqrt{\pi}}\right)\right|_{-\lambda_c}^{\infty} \\
&= \sqrt{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{\lambda_c}{\sqrt{\pi}}\right)\right).
\end{aligned}$$

(c) Determine the full-width-half-maximum width of the spectrum in part (b).

Solution

To determine the full-width-half-maximum width, solve $S_\lambda(\lambda) = \frac{\pi}{(\lambda-\lambda_c)^2+\pi} = \frac{1}{2}$ to give $\lambda = \lambda_c \pm \sqrt{\pi}$. Then the full-width-half-maximum width $= 2\sqrt{\pi}$.

(d) Estimate the coherence timewidth τ_c for the spectrum in part (b).

Solution

$\Delta t \approx 1/\Delta f = \lambda^2/c\Delta\lambda$ so that $\tau_c = \frac{\lambda^2}{2c\sqrt{\pi}}$ where $\Delta\lambda$ is the full-width-half-maximum width value of $S_\lambda(\lambda)$.

2.22 Autocorrelation and the power density spectrum of a random signal using sinusoidal pulses

A binary waveform consists of a random and independent sequence of copies of the pulse $\big(1+\cos(2\pi t/T)\big)\text{rect}(t/T)$ with random amplitude $\underline{A}_n$ for the nth term of the sequence. The start time $\underline{j}$ of the pulse sequence is a uniformly-distributed random variable over $[0,T]$. The symbols transmitted in each nonoverlapping interval of length T are independent. The probability of transmitting a mark with an amplitude A is $1/2$. The probability of transmitting a space with an amplitude 0 is $1/2$.

(a) Determine the autocorrelation function of the signal.

Solution

The form of solution follows the example shown in Figure 2.10 with the retangular pulse replaced by $\big(1+\cos(2\pi t/T)\big)$. The convolution $y(t)$ is evaluated as

$$y(t) = \begin{cases} \int_{-T/2}^{t+T/2}\big(1+\cos(2\pi\tau/T)\big)\big(1+\cos(2\pi(t-\tau)/T)\big)\mathrm{d}\tau & -T<t<0 \\ \int_{t-T/2}^{T/2}\big(1+\cos(2\pi\tau/T)\big)\big(1+\cos(2\pi(t-\tau)/T)\big)\mathrm{d}\tau & 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

which gives

$$y(t) = \begin{cases} -(1/4\pi)3T\sin(2\pi t/T) + ((t+T)/2)\cos(2\pi t/T) + t + T & -T<t<0 \\ (1/4\pi)3T\sin(2\pi t/T) - ((t-T)/2)(\cos(2\pi t/T)+2) & 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

(b) Determine the power density spectrum of the signal.

Solution

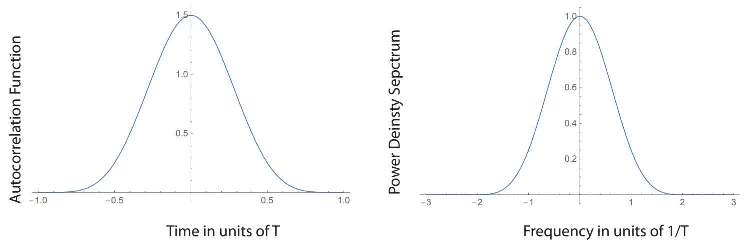
The power density spectrum is the Fourier transform of the autocorrelation function. Let $y_1(t)$ be the expression for $-T<t<0$ and let $y_2(t)$ be the expression for $0<t<T$ given in the expression listed in part (a). Then setting $T=1$ for simplicity gives

$$Y(f) = \int_{-1}^{0} y_1(t)e^{\mathrm{i}2\pi ft}\mathrm{d}t + \int_{0}^{1} y_2(t)e^{\mathrm{i}2\pi ft}\mathrm{d}t.$$

Evaluating the integrals separately and then combining gives

$$Y(f) = \frac{\sin^2(\pi f)}{\pi^2 f^2\left(f^2-1\right)^2}.$$

A plot of $y(t)$ and $Y(f)$ is given below.



2.23 *Covariance matrices*

Define $\underline{\mathbf{z}}$ as a vector of N circularly-symmetric gaussian random variables with a complex covariance matrix $\mathbb{V}$ given in (2.2.30b). Define $\underline{\mathbf{x}}$ as a vector of length $2N$ that consists of the real part Re[$\underline{\mathbf{z}}$] and the imaginary part Im[$\underline{\mathbf{z}}$] in the order $\mathbf{x} = \{\mathrm{Re}[\underline{z}_1], ..., \mathrm{Re}[\underline{z}_N], \mathrm{Im}[\underline{z}_1], ..., \mathrm{Im}[\underline{z}_N]\}$.

Show that the real $2N \times 2N$ covariance matrix $\mathbb{C}$ given by (cf. (2.2.22))

$$\mathbb{C} \;=\; \left\langle (\underline{\mathbf{x}} - \langle \mathbf{x} \rangle) \, (\underline{\mathbf{x}} - \langle \mathbf{x} \rangle)^T \right\rangle,$$

where $\underline{\mathbf{x}}$ is a random column vector formed by pairwise terms can be expressed in block form in terms of the $N \times N$ complex covariance matrix $\mathbb{V}$ as

$$\mathbb{C} \;=\; \frac{1}{2} \begin{bmatrix} \mathrm{Re}\,\mathbb{V} & -\mathrm{Im}\,\mathbb{V} \\ \mathrm{Im}\,\mathbb{V} & \mathrm{Re}\,\mathbb{V} \end{bmatrix}.$$

Solution

Let the column vector $\mathbf{x}$ of length $2N$ consist of the real part Re[$\mathbf{z}$] and the imaginary part Im[$\mathbf{z}$] in the order $\mathbf{x} = \{\mathrm{Re}[z_1], ..., \mathrm{Re}[z_N], \mathrm{Im}[z_1], ..., \mathrm{Im}[z_N]\}$. For example let $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$. Now write $\mathbb{C}$ as

$$\mathbb{C} = \langle \mathbf{x}\mathbf{x}^T \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & y_1 & y_2 \end{bmatrix} \right\rangle = \begin{bmatrix} \langle \mathrm{Re}[\mathbf{z}]\mathrm{Re}[\mathbf{z}]^T \rangle & \langle \mathrm{Re}[\mathbf{z}]\mathrm{Im}[\mathbf{z}]^T \rangle \\ \langle \mathrm{Im}[\mathbf{z}]\mathrm{Re}[\mathbf{z}]^T \rangle & \langle \mathrm{Im}[\mathbf{z}]\mathrm{Im}[\mathbf{z}]^T \rangle \end{bmatrix}$$

where, for example

$$\langle \mathrm{Re}[\mathbf{z}]\mathrm{Re}[\mathbf{z}]^T \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \right\rangle = \begin{bmatrix} \langle x_1 x_1 \rangle & \langle x_1 x_2 \rangle \\ \langle x_2 x_1 \rangle & \langle x_2 x_2 \rangle \end{bmatrix}.$$

The four 2×2 blocks make up the complete 4×4 matrix. Now use the following expressions, which can be directly verified

$$\begin{aligned}
\langle \mathrm{Re}[\mathbf{z}]\mathrm{Re}[\mathbf{z}]^T\rangle &= \frac{1}{2}\mathrm{Re}\mathbb{V} \\
\langle \mathrm{Re}[\mathbf{z}]\mathrm{Im}[\mathbf{z}]^T\rangle &= -\frac{1}{2}\mathrm{Im}\mathbb{V} \\
\langle \mathrm{Im}[\mathbf{z}]\mathrm{Re}[\mathbf{z}]^T\rangle &= \frac{1}{2}\mathrm{Im}\mathbb{V} \\
\langle \mathrm{Im}[\mathbf{z}]\mathrm{Im}[\mathbf{z}]^T\rangle &= \frac{1}{2}\mathrm{Re}\mathbb{V}
\end{aligned}$$

For example,

$$\begin{aligned}
\mathbb{V} = \langle \mathbf{z}\mathbf{z}^\dagger\rangle &= \left\langle \begin{bmatrix} x_1 + \mathrm{i}y_1 \\ x_2 + \mathrm{i}y_2 \end{bmatrix} \begin{bmatrix} x_1 - \mathrm{i}y_1 & x_2 - \mathrm{i}y_2 \end{bmatrix} \right\rangle \\
&= \begin{bmatrix} \langle x_1x_1 + y_1y_1\rangle & \langle x_1x_2 + y_1y_2\rangle - \mathrm{i}\langle \mathrm{x_1y_2 - x_2y_1}\rangle \\ \langle x_1x_2 + y_1y_2\rangle + \mathrm{i}\langle \mathrm{x_1y_2 - x_2y_1}\rangle & \langle x_2x_2 + y_2y_2\rangle \end{bmatrix}.
\end{aligned}$$

When all of the variances are equal, $\langle x_1x_1 + y_1y_1\rangle = 2\langle x_1x_1\rangle$. This is the origin of the factor of one half. Substituting gives the desired expression

$$\mathbb{C} \quad = \quad \frac{1}{2}\begin{bmatrix} \mathrm{Re}\,\mathbb{V} & -\mathrm{Im}\,\mathbb{V} \\ \mathrm{Im}\,\mathbb{V} & \mathrm{Re}\,\mathbb{V}. \end{bmatrix}.$$

2.25 *Diagonalizing a covariance matrix*

A real covariance matrix $\mathbb{C}$ of a bivariate gaussian random variable is given by

$$\mathbb{C} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}.$$

(a) Determine a new coordinate system (x', y') such that the joint probability density function in that coordinate system is a product distribution and express the joint probability density function in that new coordinate system as the product of two one-dimensional gaussian probability density functions.

Solution

The inverse of $\mathbb{C}$ is

$$\mathbb{C}^{-1} = \frac{1}{3}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix},$$

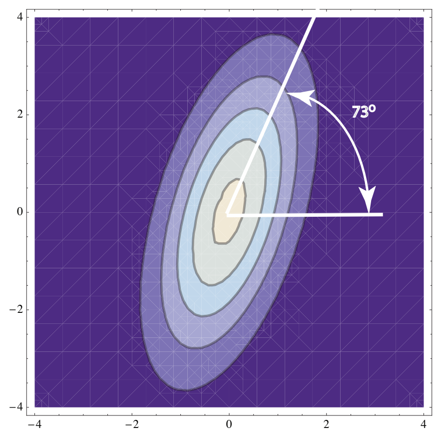
and the determinant is equal to 3. Using these expressions, the joint distribution is

$$f_{\underline{\mathbf{x}}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\sqrt{\det \mathbb{C}}} e^{-\frac{1}{2}(\mathbf{x}-\langle\mathbf{x}\rangle)^T\mathbb{C}^{-1}(\mathbf{x}-\langle\mathbf{x}\rangle)}$$
$$= \frac{1}{2\pi\sqrt{3}}\, e^{-\left(4x^2-2xy+y^2\right)/6},$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Plot this probability density function using a contour plot showing the original coordinates $(x, y)$and the transformed coordinates (x', y').

Solution
A plot of the distribution is below



(c) Determine the angle θ of rotation defined as the angle between the x axis and the x' axis.

Solution
The eigenvalues of the autocovariance matrix are

$$\frac{1}{2}\left(5 \pm \sqrt{13}\right),$$

and the eigenvectors are

$$\begin{bmatrix} \frac{1}{2}\left(-3+\sqrt{13}\right) - 4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{2}\left(-3-\sqrt{13}\right) - 4 \\ 1 \end{bmatrix}.$$

The angle of the major axis is the angle that the eigenvector that corresponds to the largest eigenvalue makes with respect to the x-axis. This is given by

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\frac{1}{2}\left(-3+\sqrt{13}\right)-4}\right) = 73^o.$$

. This angle is shown in the figure.

2.33 *Square-law photodetector*

A finite energy lightwave field $U(\mathbf{r},t)$ is directly photodetected to produce an electrical waveform $r(t)$ given by

$$\begin{aligned} r(t) &= \mathcal{R}P(z,t) = \mathcal{R}\int_{\mathcal{A}} I(\mathbf{r},t)\mathrm{d}A \\ &= \mathcal{R}\int_{\mathcal{A}} |\mathbf{U}(\mathbf{r},t)|^2\,\mathrm{d}A, \end{aligned}$$

where (1.2.4) has been used.

(a) Show that when $\mathbf{U}(\mathbf{r},t)$ is bandlimited to the frequency interval $-W \le f \le W$, the electrical waveform $r(t)$ is bandlimited to the interval $-2W \le f \le 2W$.

Solution
The intensity is defined as the square of the lightwave field. The Fourier transform of the intensity is the convolution of the spectrum of the lightwave field with the complex conjugate of the Fourier transform of the lightwave field. Because the Fourier transform is bandlimited to the frequency interval $-W \le f \le W$, $r(t)$, the convolution of the Fourier transform $S(\omega)$ of $s(t)$ with the Fourier transform $S^*(\omega)$ of $s^*(t)$. The support of this convolution in the frequency domain is twice that of $S(\omega)$. Therefore, the Fourier transform is bandlimited to the interval $-2W \le f \le 2W$.

(b) Given the coherence timewidth τ_c of the lightwave signal, estimate the coherence timewidth of the directly photodetected electrical signal.

Solution
When the coherence timewidth τ_c of the lightwave signal is expressed using a root-mean squared value, the squaring operation of direct photodetection in the time domain corresponds to a convolution in the frequency domain as was discussed in part (a). The mean-squared bandwidth (or variance) then doubles (See. Problem 2.6) with the root mean-squared bandwidth increased by $\sqrt{2}$. Using the approximate reciprocal relationship between the bandwidth and the coherence time, the root-mean-squared coherence timewidth

of the directly photodetected lightwave signal is reduced by a factor of $\sqrt{2}$.

(c) Estimate the associated width of the lightwave signal power density spectrum and the width of the electrical power density spectrum.

Solution

In the frequency domain, the spectrum is convolved with itself and thus the mean-squared bandwidths add (See. Problem 2.6) with the root mean-squared bandwidth of the directly photodetected lightwave signal increased by $\sqrt{2}$. The electrical autocorrelation function is defined using a product of the directly photodetected lightwave signal and a delayed copy of that signal. Accordingly, there is an additional factor of $\sqrt{2}$. Therefore, the root-mean-squared width of the electrical autocorrelation function is approximately a factor of two less than the root-mean-squared coherence time of the lightwave signal incident to the photodetector because of the two squaring operations.

Chapter 3 - Selected Solutions

3.1 *Coupling efficiency into a fiber*

(a) Suppose that the radiation emitted by a lightwave source is conical, independent of ϕ, and has a small numerical aperture. Show that the solid angle Ω subtended by the lightwave source is given by $\Omega \approx \pi \mathrm{NA}^2$, where NA is the numerical aperture.

Solution

The solid angle Ω is defined by

$$\begin{aligned}
\Omega &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\theta_{\max}} \sin\theta \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\arcsin(\mathrm{NA})} \sin\theta \, \mathrm{d}\theta \\
&= -\, 2\pi \cos\theta \big|_0^{\arcsin(\mathrm{NA})} \\
&= 2\pi\big(1 - \sqrt{1-\mathrm{NA}^2}\big),
\end{aligned}$$

where $\cos \mathrm{NA} = \sqrt{1-\mathrm{NA}^2}$ has been used. Expanding $\sqrt{1-\mathrm{NA}^2} \approx 1 - \frac{1}{2}\mathrm{NA}^2$ and simplifying gives

$$\Omega = 2\pi\big(1 - \sqrt{1-\mathrm{NA}^2}\big) \approx 2\pi\Big[1 - \Big(1 - \frac{1}{2}\mathrm{NA}^2\Big)\Big] = \pi\mathrm{NA}^2.$$

(b) A source emits light with a power P and with an angular distribution $I(\theta) = P\cos\theta/\pi$ where $I(\theta)$ is the power per solid angle (with units of Watts/sr) in the direction θ. Show that the coupling efficiency into the fiber is equal to NA^2.

Solution

The total power subtended by a cone with a solid angle Ω defined by a axial angle θ is shown in Figure 3.5 where θ is the maximum acceptance angle into the fiber and is given by θ=arcsin NA from (3.2.4). The total power collected by the fiber within the solid angle Ω is

$$\begin{aligned}
P &= \int_\Omega P(\Omega)\mathrm{d}\Omega \\
&= \frac{P}{\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\arcsin(\mathrm{NA})} \sin\theta\cos\theta \, \mathrm{d}\theta \\
&= P\sin^2\theta\big|_0^{\arcsin(\mathrm{NA})} \\
&= P\,\mathrm{NA}^2
\end{aligned}$$

.

(c) The radiation pattern for many sources can be modeled as $I(\theta) = (n+1)P\cos^n\theta/2\pi$ where n is an integer. Find the coupling efficiency for a lightwave source of this form. (This should reduce to part (b) for $n = 1$.)

Solution

The integral over the solid angle Ω becomes

$$\begin{aligned} P &= \int_\Omega P(\Omega)d\Omega \\ &= \frac{P(n+1)}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\mathrm{arcsin(NA)}}\sin\theta\cos^n\theta\,\mathrm{d}\theta \\ &= -P\cos^{n+1}\theta\big|_0^{\mathrm{arcsin(NA)}} \\ &= P\big(1-(1-\mathrm{NA}^2)^{(n+1)/2}\big), \end{aligned}$$

where the cos(arcsin(NA)=$\sqrt{1-\mathrm{NA}^2}$ has been used. The expression reduces to Part (a) when $n = 1$.

3.4 *TE and TM modes*

(a) Starting with

$$\begin{aligned} \nabla\times\boldsymbol{\mathcal{E}} &= -\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t} \\ \nabla\times\boldsymbol{\mathcal{H}} &= \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t} \\ \nabla\cdot\boldsymbol{\mathcal{B}} &= 0 \\ \nabla\cdot\boldsymbol{\mathcal{D}} &= 0, \end{aligned}$$

and the constitutive relationships

$$\begin{aligned} \boldsymbol{\mathcal{B}} &= \mu_0\boldsymbol{\mathcal{H}} \\ \boldsymbol{\mathcal{D}} &= \varepsilon_0\boldsymbol{\mathcal{E}}+\boldsymbol{\mathcal{P}}, \end{aligned}$$

derive Maxwell's equations restricted to a monochromatic field given by

$$\begin{aligned} \nabla\times\mathbf{E} &= -\mathrm{i}\omega\mu_0\mathbf{H} \\ \nabla\times\mathbf{H} &= \mathrm{i}\omega\varepsilon\mathbf{E}. \end{aligned}$$

.

Solution
Substituting $\boldsymbol{\mathcal{D}} = \varepsilon \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{\mathcal{H}}$ into the two curl equations gives

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{E}} &= -\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \\ \nabla \times \boldsymbol{\mathcal{H}} &= \varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}. \end{aligned}$$

Replacing $\partial/\partial t$ with the term $\mathrm{i}\omega$, and where $\boldsymbol{\mathcal{E}} \to \mathbf{E}$ and $\boldsymbol{\mathcal{H}} \to \mathbf{H}$ for the monochromatic form of the equation gives

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mathrm{i}\omega\mu_0 \mathbf{H} \\ \nabla \times \mathbf{H} &= \mathrm{i}\omega\varepsilon \mathbf{E}. \end{aligned}$$

(b) Now suppose that $\mathbf{E}(x, y, z) = [E_x(x, y)\widehat{\mathbf{x}} + E_y(x, y)\widehat{\mathbf{y}} + E_z(x, y)\widehat{\mathbf{z}}]e^{-\mathrm{i}\beta z}$ and $\mathbf{H}(x, y, z) = [H_x(x, y)\widehat{\mathbf{x}} + H_y(x, y)\widehat{\mathbf{y}} + H_z(x, y)\widehat{\mathbf{z}}]e^{-\mathrm{i}\beta z}$. Substituting this form into the monochromatic form of (2.3.1a) and into the monochromatic form of (2.3.1b), show that each transverse field component (E_x, E_y, H_x and H_y) can be written in terms of the axial components (E_z and H_z) and thus show that a transverse electromagnetic (TEM) mode cannot propagate in a dielectric slab waveguide.

Solution
The curl operation in rectangular coordinates is

$$\nabla \times \mathbf{E} \quad = \quad \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\mathrm{i}\omega\mu_0 \mathbf{H}.$$

Noting that $\partial/\partial z \to -\mathrm{i}\beta$ for the form of the electromagnetic fields given in the problem we have

$$\frac{\partial E_z}{\partial y} + \mathrm{i}\beta E_y = -\mathrm{i}\omega\mu_0 H_x \tag{1a}$$

$$-\frac{\partial E_z}{\partial x} - \mathrm{i}\beta E_x = -\mathrm{i}\omega\mu_0 H_y \tag{1b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mathrm{i}\omega\mu_0 H_z \tag{1c}$$

The second curl equation gives

$$\frac{\partial H_z}{\partial y} + \mathrm{i}\beta H_y = \mathrm{i}\omega\varepsilon E_x \tag{2a}$$

$$-\frac{\partial H_z}{\partial x} - \mathrm{i}\beta H_x = \mathrm{i}\omega\varepsilon E_y \tag{2b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \mathrm{i}\omega\varepsilon E_z \tag{2c}$$

Solve for H_x in (1a)

$$H_x = \frac{\mathrm{i}}{\omega\mu_0}\left(\frac{\partial E_z}{\partial y} + \mathrm{i}\beta E_y\right).$$

Solve for E_y in (2b) and substitute into the preceding equation

$$H_x = \frac{\mathrm{i}}{\omega\mu_0}\left[\frac{\partial E_z}{\partial y} + \mathrm{i}\beta\frac{1}{\mathrm{i}\omega\varepsilon}\left(-\frac{\partial H_z}{\partial x} - \mathrm{i}\beta H_x\right)\right].$$

Combining terms we have

$$H_x = \frac{\mathrm{i}}{k_0^2 n^2 - \beta^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right),$$

where $k_0^2 n^2 = \omega^2\varepsilon\mu_0$. The expression shows that the transverse component H_x can be expressed in terms of the two axial components E_z and H_z. Each of the other transverse components can also be expressed in terms of the axial components E_z and H_z. Therefore, if both E_z and H_z and zero, then all of the components are zero. This means that a TEM mode, for which $E_z = 0$ and $H_z = 0$ cannot be supported in a dielectric waveguide.

3.5 *Boundary conditions for TE and TM modes*

(a) Specialize Maxwell's equations to a monochromatic field propagating in the z direction when all field components have a dependence of the form $e^{\mathrm{i}(\omega t - \beta z)}$.

Solution

The governing equations for monochromatic fields are

$$\nabla \times \mathbf{E} = -\mathrm{i}\omega\mu_0\mathbf{H}$$

$$\nabla \times \mathbf{H} = \mathrm{i}\omega\epsilon\mathbf{E}.$$

Combining these equations produces the Helmholtz equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} \quad = \quad 0.$$

Given that both fields are of the form $e^{-\mathrm{i}\beta z}$, $\nabla\mathbf{E} = \nabla_t\mathbf{E} - \beta^2\mathbf{E}$ where the t refers to the transverse components. The resulting Helmholtz equation is then

$$\nabla_t^2\mathbf{E} + k_t^2\mathbf{E} \quad = \quad 0,$$

where $k_t^2 = k^2 - \beta^2$.

(b) Now suppose that $\mathbf{E} = E_y(x, z)\widehat{\mathbf{y}}$, as was the case for the slab waveguide. Determine the relationship between E_y and H_z, and thus show that H_z is proportional to $\mathrm{d}E_y/\mathrm{d}x$.

Solution

Using $\nabla \times \mathbf{E} = -\mathrm{i}\omega\mu_0\mathbf{H}$, the governing equation relating $H_z$and E_y is

$$\frac{\mathrm{d}E_y}{\mathrm{d}x} - \frac{\mathrm{d}E_x}{\mathrm{d}y} \quad = \quad -\mathrm{i}\omega\mu_0 H_z. \tag{3}$$

Therefore, when $E_x = 0$ as in a slab waveguide, $H_z \propto \mathrm{d}E_y/\mathrm{d}x$.

(c) Repeat part (b) if $\mathbf{H} = H_y(x, z)\widehat{\mathbf{y}}$ and show that E_z is proportional to $(n_2/n_1)^2\, \mathrm{d}H_y/\mathrm{d}x$.

Solution

Using $\nabla \times \mathbf{H} = \mathrm{i}\omega\epsilon\mathbf{E}$, and setting $H_x = 0$ as in a slab waveguide, the governing equation relating $E_z$and $H_y$is

$$\frac{\mathrm{d}H_y}{\mathrm{d}x} \quad = \quad \mathrm{i}\omega\epsilon_i E_z, \tag{4}$$

where $\epsilon_1 = \epsilon_0 n_1^2$ in the core and $\epsilon_2 = \epsilon_0 n_2^2$ in the cladding. Because the index is different in the two regions, in order for E_z to be continuous across the boundary requires that $E_z \propto (n_2/n_1)^2\, \mathrm{d}H_y/\mathrm{d}x$.

(d) Explain why there is a difference in the boundary conditions between the TE and TM modes for dielectric materials.

Solution

For a TE mode given in (3) the fields are scaled by μ_0, which is a constant that does not change between the waveguiding regions. For the TM mode given in (4), the fields are related by ϵ_i, which does change between the waveguiding regions because the index changes. This produces the additional term for the TM mode.

3.6 *Derivation of the Bessel differential equation*

Starting with the form of the Helmholtz equation given in (3.3.19) in cylindrical coordinates and trying a solution of the form $E_z(r, \psi, z) = f(r) \exp\left[-\mathrm{i}(\nu\psi + \beta z)\right]$, use the separation of variables method to derive the Bessel differential equation (cf. (3.3.20)).

Solution

The Laplacian in cylindrical coordinates is

$$\nabla^2 E_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \psi^2} + \frac{\partial^2 E_z}{\partial z^2}.$$

Now write $E_z(r, \psi, z)$ as a product of three functions

$$E_z(r, \psi, z) = f(r)\Psi(\psi)Z(z),$$

where $\Psi(\psi)Z(z) = e^{-\mathrm{i}(\nu\psi+\beta z)}$. Using the form of the Laplacian and $E_z(r, \psi, z) = f(r)e^{-\mathrm{i}(\nu\psi+\beta z)}$ substitute this expression into the Helmholtz equation given by

$$\nabla^2 E_z(r, \psi, z) + n^2 k_0^2 E_z(r, \psi, z) = 0,$$

to yield

$$\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} + \left(n^2 k_0^2 - \beta^2 - \frac{\nu^2}{r^2}\right) f(r) = 0,$$

which is (3.3.20).

3.7 *Normalized frequency*

A fiber has the following specifications: index of refraction $n = 1.46$, normalized index difference $\Delta = 0.0036$, and core diameter $d = 8.3$ microns.

(a) Derive the expression for the normalized frequency V in terms of:

(i) The numerical aperture.
(ii) The index difference.

Solution

(i) The expression for the normalized frequency V as a function of the NA is (cf. (3.3.12a))

$$V = a\frac{2\pi}{\lambda_c}\sqrt{(n_1^2 - n_2^2)} = a\frac{2\pi}{\lambda_c}\mathrm{NA}.$$

(ii) Rewriting the NA in terms of the index difference using (3.2.6) gives (cf. (3.3.12b))

$$V \quad = \quad a\frac{2\pi}{\lambda_c}\mathrm{NA} \quad = \quad a\frac{2\pi}{\lambda_c}n_1\sqrt{2\Delta},$$

which is (3.3.12b) with $2\pi/\lambda_c = k_0$.

(b) For a fiber that has a core index of 1.5 and $\Delta = 0.1\%$, what is the largest core that can support single-mode operation at a wavelength of $1.3\,\mu$m?

Solution

For single mode operation, V_{max}=2.4. Solving for the radius a from the expression derived in part (a) yields

$$\begin{aligned} a &= V_{\mathrm{max}}\frac{\lambda_c}{2\pi n_1\sqrt{2\Delta}} \\ &= 2.4\frac{1.3}{2\pi \times 1.5\sqrt{2\times 10^{-3}}} \quad = \quad 7.4\,\text{microns} \end{aligned}$$

3.11 *Linearly-polarized modes of a fiber (requires numerics.)*

The normalized frequency V of a step-index fiber is 4.

(a) Using the mode characteristics of a linearly-polarized mode given in Figure 3.15, determine which modes are guided in the fiber, and estimate the normalized propagation constant b (cf. (4.2.1)) for each guided mode.

Solution

Examining Figure 3.15 for $V = 4$, there are four modes that are guided. LP_{01}, LP_{11}, LP_{21} LP_{02}. Using (4.2.1), which relates pa, qa and V gives:

For the LP_{01} mode, b= 0.77
For the LP_{11} mode, b= 0.44
For the LP_{21} mode, b= 0.047
For the LP_{02} mode, b= 0.00446

(b) For the two modes with the largest values of b, use a root-finding algorithm to numerically find the values of pa, qa, and b.

Solution

LP_{01} mode, b=0.77 (0.7727), $qa = 3.51$, $pa = 1.9$.
LP_{11} mode, b = 0.44 (0.440), $qa = 2.65$, $pa = 2.99$.

As can be seen, for $b \longrightarrow 1$, the mode is well guided, $qa \longrightarrow V$ and $pa \longrightarrow 0$. For modes near cutoff, $b \longrightarrow 0$, $qa \longrightarrow 0$, and $pa \longrightarrow V$.

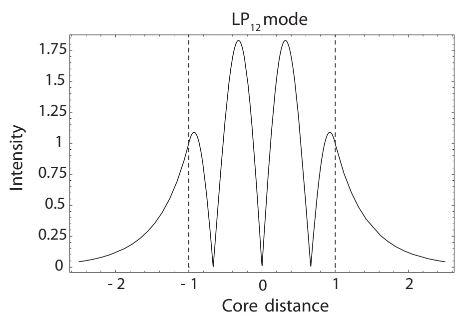
(c) Using (3.3.45), plot the radial dependence of the intensity of the field for the mode closest to cutoff.

Solution

For $V = 4$, the mode closest to cut-off is the LP_{02} mode. The values for this mode are: b = 0.004459, qa = 0.267, and pa=3.991, which is nearly equal to V. Using (3.3.45), the form for the radial dependence is

$$\frac{J_\nu(pr)}{J_\nu(pa)} \text{ for } r \leq a \qquad\qquad \frac{K_\nu(qr)}{K_\nu(qa)} \text{ for } r \geq a.$$

A plot of the intensity is shown below.



3.18 *Modes of an infinite parabolic-profile graded-index fiber*

Let the inhomogeneous index of refraction profile for a graded-index fiber be given as

$$n^2(x,y) \quad = \quad n_0^2\left[1 - 2\Delta\left(\frac{x^2+y^2}{a^2}\right)\right],$$

where a is the core radius.

(a) Assuming a solution of the form

$$U(\mathbf{r}) \quad = \quad U(x,y,z) \quad = \quad AU(x,y)e^{-\mathrm{i}\beta z},$$

where A is a constant, substitute this form of solution and the index of refraction profile given above into the scalar Helmholtz equation

$$\nabla^2 U(x,y,z) + n(r)^2 k_0^2 U(x,y,z) \quad = \quad 0,$$

and show that

$$\frac{\partial^2 U(\mathbf{r})}{\partial x^2} + \frac{\partial^2 U(\mathbf{r})}{\partial y^2} + \left[k_0^2 n_0^2 \left(1 - 2\Delta \left(\frac{x^2}{a^2} + \frac{y^2}{a^2}\right)\right) - \beta^2\right] U(\mathbf{r}) \quad = \quad 0.$$

Solution
Using $\partial^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $\partial^2/\partial z^2 \to -\beta^2$ gives the desired equation.

(b) Using the separation of variables method with $U(x,y) = f(x)h(y)$ show that

$$\begin{aligned} \frac{1}{f(x)} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} - \left(\frac{2k_0^2 n_0^2 \Delta}{a^2}\right) x^2 &= -K_1 \\ \frac{1}{h(y)} \frac{\mathrm{d}^2 h(y)}{\mathrm{d}y^2} - \left(\frac{2k_0^2 n_0^2 \Delta}{a^2}\right) y^2 &= -K_2, \end{aligned}$$

where K_1 and K_2 are two separation constants.

Solution
Writing $U(x,y) = f(x)h(y)$, using $\partial U(x,y)/\partial x = h(y)\mathrm{d}f(x)/\mathrm{d}x$, $\partial U(x,y)/\partial y = f(x)\mathrm{d}h(y)/\mathrm{d}y$ and dividing through by $f(x)h(y)$ gives the desired equations.

(c) Show that the two separation constants satisfy

$$\beta^2 \quad = \quad k_0^2 n_0^2 - K_1 - K_2.$$

Solution
See the solution to part (b).

Chapter 4 Selected Solutions

4.1 *Transit time delay using ray optics and the equivalent frequency transfer function*

The maximum delay spread in ray optics is the difference between the delay of the ray that takes the longest time and the delay of the ray that takes the shortest time to travel the same distance in a fiber. A distribution of delays results from a distribution of rays coupled into the fiber at various angles. Suppose that the propagation times associated with this distribution of rays is uniformly distributed between the limiting values of τ_1 and τ_2, where τ_2 is larger than τ_1.

(a) Determine the functional form of the distribution of the ray delays.

Solution

The distribution of the delays is uniform.

(b) Determine the root-mean-squared delay spread.

Solution

The root-mean squared width of this distribution is

$$\sigma_t^2 \quad = \quad \frac{\int_{-\infty}^{\infty}(t-\bar{t})^2 g(t)\mathrm{d}t}{\int_{-\infty}^{\infty} g(t)\mathrm{d}t} \text{where } \bar{t} \quad = \quad \frac{\int_{-\infty}^{\infty} t g(t)\mathrm{d}t}{\int_{-\infty}^{\infty} g(t)\mathrm{d}t}.$$

Because the pulse has unit height and a width of W, the area is W. Because the function is even, $\bar{t} = 0$ (the integral of an odd function $\times$ an even function over a symmetric interval is zero). Therefore,

$$\begin{aligned}
\sigma_t^2 &= \frac{\int_{-\infty}^{\infty}(t-\bar{t})^2 g(t)\mathrm{d}t}{\int_{-\infty}^{\infty} g(t)\mathrm{d}t} \\
&= \frac{1}{W}\int_{-\infty}^{\infty} t^2 g(t)\mathrm{d}t \\
&= \frac{1}{W}\int_{-\frac{W}{2}}^{\frac{W}{2}} t^2 \mathrm{d}t \quad = \quad \frac{W^2}{12} \\
&\rightarrow \quad \sigma_t = \frac{W}{\sqrt{12}}.
\end{aligned}$$

(c) Now suppose that the distribution of delay times determined in part (a) is used to model the impulse response $h(t)$ for the fiber. Determine the frequency response $H(\omega)$ in terms of the differential delay $\tau_2 - \tau_1$.

Solution
The transfer function $H(f)$ is the Fourier transform of the impulse response $h(t)$ which is the Fourier transform of the rect function so that $H(f)$ is equal to $\text{sinc} f$.

(d) A lightwave source is characterized by a numerical aperture NA_s that is smaller than the numerical aperture NA of the fiber. Two rays coupled by this source into the fiber are to be compared. One ray along the axis and one ray defined by NA_s where NA_s is much smaller than one. Determine the ratio of the differential time delay using this lightwave source relative to the differential time delay using a different lightwave source with a numerical aperture equal to the numerical aperture of the fiber.

Solution
Solving for Δ in terms of the NA gives

$$\Delta \quad = \quad \frac{1}{2}\frac{\text{NA}^2}{n_1^2}.$$

Substituting Δ into the expression for the transit time spread, the differential time delay can be written as

$$\tau \quad = \quad \tau_2 - \tau_1 \quad = \quad \frac{Ln_1}{c}\Delta \quad = \quad \frac{1}{2}\frac{\text{NA}^2}{n_1}\frac{L}{c}.$$

In the same way, the transit time spread τ_s for the second lightwave source can be written as

$$\tau_s \quad = \quad \tau_2 - \tau_1 \quad = \quad \frac{Ln_1}{c}\Delta_s$$

where

$$\Delta_s \quad \doteq \quad \frac{1}{2}\frac{\text{NA}_s^2}{n_1^2}.$$

The ratio of the transit times is

$$\frac{\tau}{\tau_s} \quad = \quad \frac{1}{2}\frac{\text{NA}^2}{n_1^2}\Big/\frac{1}{2}\frac{\text{NA}_s^2}{n_1^2} \quad = \quad \frac{\text{NA}^2}{\text{NA}_s^2}\ .$$

The transit time spread decreases as the ratio of the squares of the numerical apertures. This statement emphasizes that for a fiber that can support multiple rays (or modes), the transit time spread and the associated bandwidth depends on the distribution of the launched rays as measured by the source NA_s. The dependence of the fiber response as measured by τ on the launch conditions, as measured by NA_s can lead to a launch-dependent channel response. This dependence must be controlled for reliable communications.

4.5 *Mode-group density*

The *mode-group density* in an optical fiber, defined as $\Delta\beta \doteq \mathrm{d}\beta/\mathrm{d}g$, represents the closeness of the mode spacing with respect to the mode-group index g.

(a) Starting with the approximate expression for the dispersion relationship given in (4.4.4), derive $\Delta\beta$ as α approaches infinity, which corresponds to a step-index fiber.

Solution

When Δ is much less than one, the expression for $\beta(\omega, g)$ given in (4.4.4) reduces to

$$\beta(\omega, g) \quad \approx \quad n_0 k_0 \left[1 - \Delta \left(\frac{g}{\mathcal{G}}\right)^{\frac{2\alpha}{\alpha+2}}\right].$$

As α goes to infinity, this expression simplifies to

$$\beta(\omega, g) \quad \approx \quad n_0 k_0 \left[1 - \Delta \left(\frac{g}{\mathcal{G}}\right)^{2}\right].$$

The derivative of this expression with respect to the mode-group index g is

$$\Delta\beta \quad \approx \quad -2\Delta n_0 k_0 \left(\frac{g}{\mathcal{G}}\right).$$

showing that $\Delta\beta$ is linear in the mode-group index g.

(b) What is the corresponding density with respect to the mode index m?

Solution

There are approximately $2g$ modes in most mode groups. Therefore the mode density is twice the mode group density.

(c) Repeat for $\alpha = 2$. Compare the mode-group density of a step-index fiber to the mode density of a parabolic power-law graded-index fiber. Comment on the result.

Solution

For $\alpha = 2$, the exponent is equal to one and

$$\beta(\omega, g) \quad \approx \quad n_0 k_0 \left[1 - \Delta \left(\frac{g}{\mathcal{G}}\right)\right].$$

The derivative of this expression is

$$\Delta\beta \quad \approx \quad -\left(\frac{\Delta n_0 k_0}{\mathcal{G}}\right),$$

showing that to the first order of approximation, $\Delta\beta$ is a constant independent of the mode group number g so that every mode group travels with approximately the same group velocity.

4.6 *Index of refraction, group index, and material dispersion coefficient for silica glass*

An empirical expression called the *Sellmeier formula* is often used to model the index $n(\lambda)$ of glass as a function of wavelength. One form of the Sellmeier formula for silica glass is

$$n(\lambda) = \sqrt{1 + \frac{1.0955 \times 10^{18}\lambda^2}{10^{18}\lambda^2 - 100^2} + \frac{0.9 \times 10^{18}\lambda^2}{10^{18}\lambda^2 - 9000^2}}.$$

(a) Plot the material dispersion D_λ over the range of 500-1500 nm. As a check, Figure 4.6 used the same formula.

Solution

The expressions for the group index $N(\lambda)$ is given in (4.3.7) and is repeated here

$$N(\lambda) = n(\lambda) - \lambda\frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda},$$

where $n(\lambda)$ and $N(\lambda)$ are understood to mean $n(2\pi c_0/\lambda)$ and $N(2\pi c_0/\lambda)$. The expression for the dispersion coefficient D_λ is given in (4.3.8) and is repeated here

$$D_\lambda = \frac{1}{c_0}\frac{\mathrm{d}N(\lambda)}{\mathrm{d}\lambda},$$

or

$$\begin{aligned} D_\lambda &= \frac{1}{c_0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(n(\lambda) - \lambda\frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}\Big) \\ &= -\frac{\lambda}{c_0}\frac{\mathrm{d}^2 n(\lambda)}{\mathrm{d}\lambda^2}. \end{aligned} \tag{5}$$

Using the expression for the index of refraction given in the problem, a plot of the index $n(\lambda)$ and the group index $N(\lambda)$ is shown on the left side of Figure 4.6, with the group index being the curve with the minimum. The dispersion is shown on the right side of Figure 4.6, where the units are ps/(nm · km). The calculated group index at 1310 nm is 1.4613, as compared to a typical value of the group index from a data sheet of 1.4675 . This is a 0.42% difference.

(b) Determine the material dispersion minimum and the dispersion slope at the minimum.

Express the slope in units of $\mathrm{ps}/(\mathrm{nm}^2 \cdot \mathrm{km})$.

Solution

Solving for the zero crossing of the dispersion coefficient gives a minimum value of 1275 nm. The slope at the dispersion minimum is $0.125\ \mathrm{ps}/(\mathrm{nm}^2 \cdot \mathrm{km})$.

(c) What is the maximum spectral width of a pulse at 1300 nm that will limit the material dispersion to 50 ps for a fiber with length 75 km and a material dispersion $D_\lambda = 1.2$ $\mathrm{ps}/(\mathrm{nm} \cdot \mathrm{km})$?

Solution

Ignoring waveguide dispersion, rewrite the root-mean squared spectral width σ_λ (4.5.10) as

$$\sigma_\lambda = \frac{\sigma_{\text{intra}}}{LD_\lambda}.$$

Substituting the numerical values gives

$$\sigma_\lambda = \frac{50\ \text{ps}}{75\text{km} \times 1.2\text{ps}/(\text{nm} \cdot \text{km})} = 0.6\ \text{nm}.$$

4.7 *Fiber modes and dispersion*

A step-index fiber with a numerical aperture equal to 0.15 and a core index $n_1 \approx N_1 = 1.5$ operates at 850 nm and supports two modes with normalized propagation constants $b = 0.4$ and $b = 0.75$.

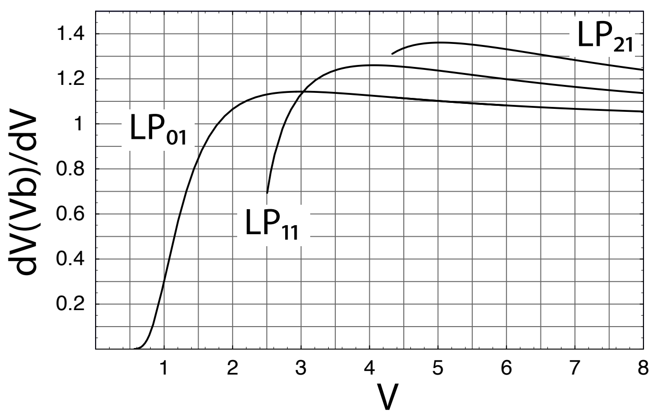
(a) What is the core diameter of the fiber?

Solution

Referring to the right side of Figure 4.3 with $b = 0.75$ for the LP_{01} mode and $b = 0.4$ for the LP_{11} mode, the approximate value of V is 3.8. Solving for the radius a gives

$$a = \frac{V\lambda_0}{2\pi\text{NA}} = \frac{3.8(0.85)}{2\pi(0.15)} = 3.43\ \mu\text{m}.$$

(b) Using the figure shown below, determine the distance into the fiber at which the modal delay between the two modes is 2.5 ns.



Solution

Referring to the figure, $d(Vb)/dV$ is estimated to be 1.13 at $V = 3.8$ for the LP_{01} mode and 1.25 for the LP_{11}mode. Using (4.4.10), the differential delay $\delta\tau/L$ per kilometer is

$$\delta\frac{\tau}{L} = \frac{n_1\Delta}{c_0}\left(\frac{d(Vb_{01})}{dV} - \frac{d(Vb_{11})}{dV}\right) = 10^{12}\frac{1.45(0.0036)}{3\times10^8}(1.25-1.13) \approx 2\ \mathrm{ns/km}$$

where the scaling factor of 10^{12} converts s/m into ns/km. To achieve a total delay of 25 ns requires about 12.5 km of fiber.

4.9 *Single-mode fiber dispersion*

A lightwave system of interest operates at 1550 nm using a single-mode step-index fiber. The transmitted lightwave signal has a spectral width of $\sigma_\lambda = 0.05$ nm and transmits a pulse with a root-mean-squared width of 100 ps. This pulse propagates 50 km in the fiber. The total intramodal dispersion coefficient D in the fiber in units of $\mathrm{ps/(nm\cdot km)}$ is modeled as

$$D = \frac{S_0}{4}\left(\lambda - \frac{\lambda_0^4}{\lambda^3}\right),$$

where $\lambda_0 = 1310$ nm is the zero dispersion wavelength and the dispersion slope parameter S_0 has units of $\mathrm{ps/(nm^2\cdot km)}$.

(a) Determine the dispersion slope S_0 parameter required to limit the root-mean-squared width of the wavelength-dependent delay distribution to 25 ps.

Solution

The total spread σ_{intra} of the pulse caused by intramodal dispersion is given by (4.5.7)

$$\sigma_{\text{intra}} = L\sigma_\lambda|D|,$$

where $|D| \doteq \sqrt{D^2}$ is the absolute value of the total group-velocity dispersion coefficient D defined in (4.4.14).

Substituting the numerical values gives

$$25 \quad = \quad 50 \times 0.05 \times \frac{S_0}{4}\left(1550 - \frac{1310^4}{1550^3}\right).$$

Solving, $S_0 = 0.0258$ ps/(nm^2·km).

(b) If $S_0 = 0$, and the system operates at λ_0, is there dispersion? Provide quantitative reasoning for your answer.

Solution

There is still dispersion from the next term in the Taylor series expansion of $\beta(\omega)$ given in (4.3.1).

4.10 *Dispersion*

(a) Using Figure 3.15, determine the number of modes that propagate at 900 nm if the fiber has a diameter of 7 microns, a numerical aperture of 0.15, and an index n=1.45.

Solution

The normalized frequency V is given by

$$0.15\left(\frac{7}{2}\right)\frac{2\pi}{0.9} \quad = \quad 3.67. \tag{6}$$

The value for the normalized index difference Δ is given by (3.2.6) with $\Delta = (\mathrm{NA}/n)^2/2 = 0.005$. Examining Figure 3.15, two modes propagate.

(b) Determine the intermodal dispersion σ_{inter}/L per unit length in units of ns/km if the lowest-order mode contains 80% of the power and the rest of the power is distributed uniformly among all modes that propagate. Use the figure provided in Problem 7 to determine the delay values.

Solution

There are two modes. The intermodal dispersion is given in (4.4.10) and repeated here

$$\tau_m \quad \approx \quad \frac{L}{c_0}\left(N_1 + n_1\Delta\frac{\mathrm{d}(Vb_m)}{\mathrm{d}V}\right),$$

Reading off curve provided in Problem 7, $d(Vb)/dV$ for the LP_{01} mode is estimated as 1.13, whereas the value for the LP_{11} mode is estimated as 1.25. The group index at 900 nm is 1.465 from Figure 4.6. Using $n_1 \approx N_1$, gives

$$\langle \tau_m^2 \rangle = \left(\frac{Ln_1}{c_0}\right)^2 \left(0.8(1 + 1.13(0.005))^2 + 0.2(1 + 1.25(0.005))\right)$$

and

$$\langle \tau_m \rangle^2 = \left(\frac{Ln_1}{c_0}\right)^2 (0.8(1 + 1.13(0.005) + 0.2(1 + 1.25(0.005))^2$$

so that

$$\begin{aligned} \sigma_{\text{inter}} &= \left(\langle \tau_m^2 \rangle - \langle \tau_m \rangle^2\right)^{1/2} \\ &= 1.17 \text{ ns/km.} \end{aligned}$$

where the factor of 10^{12} converts s/m into ns/km.

(c) Using Figure 4.6 for the group index, determine the material dispersion coefficient D_λ at 900 nm when the power-density spectrum has a spectral width of 1 GHz.

Solution

The material dispersion coefficient is given in (4.3.8) and is repeated in (5)). Using the curve on the right side of Figure 4.6, this is about $D_\lambda \approx -83$ ps/(nm $\cdot$ km) at 900 nm. To determine the dispersion, we convert the spectral width in frequency to a spectral width of 1 GHz into a wavelength spectral width σ_λ to give

$$\sigma_\lambda = \frac{\lambda^2 \Delta f}{c} = \frac{\left(0.9 \times 10^{-6}\right)^2 10^9}{3 \times 10^8} = 0.0027 \text{ nm.} \tag{7}$$

The dispersion in ps/km is then $\sigma_\lambda D_\lambda$ =0.0027 nm $\times -83$ ps/(nm $\cdot$ km) $= -0.224$ps/km.

(d) Determine the waveguide dispersion term D_{guide} for the two guided modes with the largest values of b.

Solution

Using Figure 4.9, the normalized waveguide dispersion term for the LP_{01} mode is estimated as -0.1 and that for the LP_{11} mode is estimated as 0.15. The waveguide dispersion given in (4.4.13) is then

$$\begin{aligned} \sigma_{\text{wave}} &= -\frac{n_1 \Delta}{c\lambda} V \frac{d^2(Vb)}{dV^2} \\ &= \frac{-1.45 \times 0.005}{3 \times 10^8 \times 0.9 \times 10^{-6}} \times (-0.1) \quad = \quad 2.68 \text{ ps/(nm} \cdot \text{km).} \end{aligned}$$

For the LP_{11}mode, the term is -4 ps/(nm $\cdot$ km).

(e) Determine the intramodal dispersion coefficient $D = |D_\lambda + D_{\text{guide}}|$ for the two guided modes with the largest values of b.

Solution
The total dispersion is the sum and is dominated by the material dispersion. For the LP_{01} mode, we have

$$\begin{aligned} D &= \sigma_\lambda \left| D_{\text{wav}} + D_\lambda \right| \\ &= 0.0027 \left| -83 + 2.68 \right| \quad = \quad 0.217 \text{ ps/km}. \end{aligned}$$

For the LP_{11}mode, the total dispersion is 0.235 ps/km.

4.13 *Output pulse for a gaussian power density spectrum*

Let the normalized power density spectrum of a modulated lightwave signal be

$$S_\lambda(\lambda) = \frac{1}{\sqrt{2\pi}\sigma_\lambda} e^{-(\lambda-\lambda_c)^2/2\sigma_\lambda^2},$$

as a function of the wavelength λ where the carrier wavelength is $\lambda_c = 1350$ nm. The fiber has a core diameter of 9 microns and a numerical aperture of 0.15. The transmitted pulse is a square pulse of duration $T = 200$ ps over a fiber span of length 75 km with an intramodal dispersion coefficient of $D = 8$ ps/(nm $\cdot$ km). Using Figure 4.6 for the index (or group index) and the figure provided in Problem 7 for the delay terms, determine:

(a) The normalized frequency V of the fiber.

Solution
Using (3.3.12) we obtain

$$V = \text{NA}\frac{2\pi a}{\lambda_c} = 0.15\frac{9\pi}{1.35} = \pi. \tag{8}$$

(b) The value of $s_\lambda = \mathrm{d}\tau/\mathrm{d}\lambda|_{\lambda=\lambda_c}$.

Solution
Using (4.4.5) gives the expression for the intermodal dispersion $\sigma_{\text{intra}} = \sigma_\lambda \mathrm{d}\tau/\mathrm{d}\lambda$. Combining this expression with $\sigma_{\text{intra}} = L\sigma_\lambda D$ (cf. (4.5.7)) gives

$$s_\lambda = LD,$$

where D is the total dispersion given for this problem as $D = 8\,\text{ps}/(\text{nm}\cdot\text{km})$. This value includes both waveguide and material dispersion. Multiplying by the distance gives

$$s_\lambda \;=\; 75 \times 8 \;=\; 600\,\text{ps}/\text{nm}. \tag{9}$$

(c) The total root-mean-squared width σ_{intra} of the delay spread distribution over the span length L expressed in picoseconds (cf. (4.5.7)).

Solution

The total root-mean-squared width σ_{out} at the output of the fiber span is estimated using the mean-squared timewidth of the input square pulse, which is given by $T^2/12$ (see Problem, 4.1). The mean-squared spread of the fiber impulse response is estimated as $\sigma^2_{\text{fiber}} = (\sigma_\lambda DL)^2$. Summing the mean-squared values and taking the square root provides an estimate of σ_{out} given by

$$\begin{aligned}\sigma_{\text{out}} &= \sqrt{\sigma^2_{\text{in}} + \sigma^2_{\text{fiber}}}\\ &= \sqrt{\frac{T^2}{12} + (\sigma_\lambda DL)^2}.\end{aligned}$$

As a numerical example, when σ_λ= 0.25 nm, σ_{out} is approximately 160 ps and is dominated by the spreading caused by the dispersion in the fiber.

4.15 *Optimal value for index profile*

Starting with

$$\Delta\frac{\alpha - 2 - 2y}{\alpha + 2} + \frac{\Delta^2}{2}\frac{3\alpha - 2 - 4y}{\alpha + 2} \;=\; 0,$$

show that if $\alpha > 1$ and both Δ and y are small, then the optimal power-law index profile α_{opt} is given by

$$\alpha_{\text{opt}} \;=\; 2(1 + y - \Delta).$$

Solution

Solving for α in the preceding equation we have

$$\alpha_{\text{opt}} \;=\; \frac{2(2 + 2y + \Delta + 2y\Delta)}{2 + 3\Delta}.$$

Expanding this expression in a power series gives

$$\alpha_{\text{opt}} \approx (2y+2) + \Delta(-y-2) + O\left(\Delta^2\right).$$

Because both y and Δ are small, the term $y\Delta$ can be neglected giving

$$\alpha_{\text{opt}} \approx 2(1+y-\Delta).$$

4.16 *Polarization-mode dispersion vector*

(a) Using $(\mathbb{A}\mathbb{B})^\dagger = \mathbb{B}^\dagger\mathbb{A}^\dagger$, and differentiating $\mathbb{D}\mathbb{D}^\dagger = \mathbb{I}$ with respect to ω, show that the transformation $\mathrm{i}\mathbb{D}_\omega\mathbb{D}^\dagger$ is hermitian.

Solution

Starting with $\mathbb{D}\mathbb{D}^\dagger = \mathbb{I}$ and noting that the time derivative will produce $\mathrm{i}\omega$, we have

$$\mathrm{i}\omega\mathbb{D}_\omega\mathbb{D}^\dagger + \mathrm{i}\omega\mathbb{D}\mathbb{D}_\omega^\dagger = 0$$

or

$$\mathrm{i}\omega\mathbb{D}_\omega\mathbb{D}^\dagger = -\mathrm{i}\omega\mathbb{D}\mathbb{D}_\omega^\dagger$$

Using $(\mathbb{A}\mathbb{B})^\dagger = \mathbb{B}^\dagger\mathbb{A}^\dagger$ the left side is the conjugate transpose of the right side and thus $\mathrm{i}\omega\mathbb{D}_\omega\mathbb{D}^\dagger$ is Hermitian.

(b) Using the differential relationship $\mathbb{D}(\omega + \mathrm{d}\omega) = \mathbb{D} + \mathrm{d}\omega\mathbb{D}_\omega$ and $|\det\mathbb{D}| = 1$ for a unitary matrix, show that trace of $\mathbb{D}_\omega\mathbb{D}^\dagger$ is equal to zero, which implies that the eigenvalues of $\mathbb{D}$ sum to zero.

Solution

Starting with $\mathbb{D}(\omega + \mathrm{d}\omega) = \mathbb{D} + \mathrm{d}\omega\mathbb{D}_\omega$ factor out $\mathbb{D}$ on the right side to yield

$$\mathbb{D}(\omega + \mathrm{d}\omega) = \left(\mathbb{I} + \mathrm{d}\omega\mathbb{D}_\omega\mathbb{D}^\dagger\right)\mathbb{D}$$

where $\mathbb{D}^{-1} = \mathbb{D}^\dagger$ because $\mathbb{D}$ is unitary. The determinant of the left side equals one and the determinant of the first time on the right side equals one. Therefore the sides are equal if and only if the trace of $D_\omega\mathbb{D}^\dagger$ is zero, which is evident from (4.6.10).

4.19 *Dispersion relationship from ray optics*

Modes in a slab waveguide can be intuitively reconciled with ray theory by letting the ray define the direction of a plane wave propagating in the slab waveguide. In this reconciliation, a mode is formed by the interference with itself of a propagating plane wave that "zig-zags" between the core/cladding interfaces. The direction of the plane wave is shown as the line in Figure 3.10. This arrow defines a ray associated with the plane wave. The z components of the two interfering plane waves produce a traveling wave while the transverse components add to produce a standing wave.

(a) Using $k_x = p$ and $k_z = \beta$ rewrite

$$p^2 + \beta^2 \quad = \quad (n_1 k_0)^2$$

in terms of the components k_x and k_z of the wavevector **k** in the slab waveguide.

Solution
Rewriting this equation, we have

$$k_x^2 + k_z^2 \quad = \quad (n_1 k)^2,$$

where $k_x = p$ and $k_z = \beta$ are now written as the components of the wavevector of a plane wave.

(b) Derive an expression for the angle θ that the plane wave makes with respect to the normal of the core/cladding interface in terms of k_x, n, and k_0.

Solution
The angle θ the plane wave makes with the core/cladding interface is related to the x-component of the wavevector by $\cos\theta = k_x/(n_1 k)$.

(c) Consider a slab waveguide that supports a plane wave with a polarization that is transverse to the direction of propagation. Upon reflection from the core/cladding interface, a consequence of Maxwell's equations is that this plane wave experiences a phase shift ϕ_{TE} given by

$$\phi_{TE} \quad = \quad -2\arctan\left(\sqrt{\sin^2\theta_i - (n_2/n_1)^2}\Big/\cos\theta_i\right),$$

where θ is the angle from the normal to the core/cladding interface. Using this expression, determine the total phase shift the plane wave experiences after two reflections consisting of one reflection from each boundary of the slab waveguide.

Solution
After a reflection from each boundary, the plane wave experiences a phase shift of $4ak_x$ along the propagation direction z. Each reflection adds an additional phase shift ϕ_{TE} given above. For a mode to be generated, the total phase shift must be $m2\pi$ where m is an integer so that the field adds constructively with itself after the two reflections. This condition is given as

$$4ak_x + 2\phi_{\mathrm{TE}} \quad = \quad m2\pi. \tag{10}$$

(d) A guided mode in the slab waveguide is generated whenever the total phase shift is $m2\pi$ where m is an integer so that the field adds constructively with itself after the two reflections. Derive this condition and show that

$$\sqrt{(1-\cos^2\theta)-(n_2/n_1)^2}\Big/\cos\theta \quad = \quad \tan\left(n_1 k_0 a\cos\theta - m\pi/2\right).$$

Solution

Substituting the phase shift ϕ_{TE} from the reflection at the boundary into (10) along with $k_x = n_1 k\cos\theta$ and taking the tangent of each side gives

$$\sqrt{(1-\cos^2\theta)-\left(\frac{n_2}{n_1}\right)^2}\Big/\cos\theta \quad = \quad \tan\left(n_1 k a\cos\theta - \frac{m\pi}{2}\right). \tag{11}$$

Values of θ that satisfy this equation are the allowed angles for the rays that produce a self-consistent phase after two reflections. Each angle defines a mode with a corresponding propagation constant β given by (3.3.10a).

4.20 *Bandwidth-dependent launch conditions*

Consider a uniform mode distribution in which the power is uniformly distributed among all the modes in a fiber. The fraction of the power in each mode is $F_m = 1/M$, where M is the number of modes. The expressions for the group-delay terms become

$$\langle\underline{\tau}\rangle \quad = \quad \frac{1}{M}\sum_{m=1}^{M}\tau_m \qquad\qquad \langle\underline{\tau}^2\rangle \quad = \quad \frac{1}{M}\sum_{m=1}^{M}\tau_m^2,$$

where M is the number of modes. Suppose that a fiber has the following parameters: $V = 5$, $n_1 = 1.46$, $\Delta = 0.0036$, and $N_1 = 1.48$.

(a) Determine the delay spread σ_{inter} for the case of a uniform mode distribution across the LP_{01}, LP_{11} modes. Use the figure provided in Problem 7 to determine τ_m for each mode.

Solution

The expression for the delay is given in (4.4.10)

$$\tau_m \quad = \quad \frac{L}{c_0}\left[n_1\Delta\frac{d(Vb_m)}{dV} + N_1\right]$$

Using the figure provided in Problem 7, the values of $d(Vb_m)/dV$ for each of the three modes and the delay are

$$\begin{aligned} d(Vb_m)/dV \text{ for LP}_{01} &= 1.1 & \tau &= 4.953\ \mu\text{s/km} \\ d(Vb_m)/dV \text{ for LP}_{11} &= 1.24 & \tau &= 4.955\ \mu\text{s/km} \\ d(Vb_m)/dV \text{ for LP}_{21} &= 1.35 & \tau &= 4.957\ \mu\text{s/km} \end{aligned}$$

The root-mean squared spread for the uniform mode launch with the power in each mode being 1/3 is

$$\begin{aligned} \sigma_{\text{inter}} &= \left(\langle\tau_m^2\rangle - \langle\tau_m\rangle^2\right)^{1/2} \\ &= \left[\frac{1}{M}\sum_{m=1}^{M}\tau_m^2 - \left(\frac{1}{M}\sum_{m=1}^{M}\tau_m\right)^2\right]^{1/2} \\ &= \left[\frac{1}{3}(4.953^2 + 4.955^2 + 4.957^2) - \left[(4.953 + 4.955 + 4.957)/3\right]^2\right]^{1/2} \\ \sigma_{\text{inter}} &= 2.66\,\text{ps/km} \end{aligned}$$

(b) Determine the delay spread σ_{inter} when the power in the LP_{11} mode is half the power in the LP_{01} mode and the power in the LP_{21} mode is half the power in the LP_{11} mode.

Solution

The root-mean squared spread for the nonuniform mode launch with powers of 4/7, 2/7 and 1/7 in each mode is

$$\begin{aligned} \sigma_{\text{inter}} &= \left(\langle\tau_m^2\rangle - \langle\tau_m\rangle^2\right)^{1/2} \\ &= \left[\frac{1}{M}\sum_{m=1}^{M}\tau_m^2 - \left(\frac{1}{M}\sum_{m=1}^{M}\tau_m\right)^2\right]^{1/2} \\ &= \left[\frac{4}{7}4.953^2 + \frac{3}{7}4.955^2 + \frac{1}{7}4.957^2 - \left(\frac{4}{7}4.953 + \frac{3}{7}4.955 + \frac{1}{7}4.957\right)^2\right]^{1/2} \\ \sigma_{\text{inter}} &= 2.12\,\text{ps/km} \end{aligned}$$

(c) Which launch condition produces the smallest value of σ_{inter}? Why?

Solution

The non-uniform launch produces the least spread because there is a smaller proportion of the power in the higher-order modes.

4.23 *Wavelength-dependent group delay*

Refer to the figure Figure 4.9, which shows the group delay factor for two linearly-polarized modes.

(a) Find the value of the normalized frequency V for which the group delay term for the LP_{01} mode is equal to the group delay term for the LP_{11} mode.

Solution
Examining the figure, the value of the normalized frequency V is estimated to be 3.

(b) For this value of V, is the group velocity dispersion coefficient the same for each mode? Explain.

Solution
No. The slope at this point, which is proportional to the group-velocity dispersion coefficient (cf. (4.3.2c)) is not the same for each mode.

(c) Over the range of values shown in Figure 4.9, is there a value of V for which the group velocity dispersion of the LP_{01} mode is equal to the group velocity dispersion of the LP_{11} mode?

Solution
This is equivalent to asking if slope is the same for a given value of the normalized frequency V. For the range of values of V shown in Figure 4.9, there is no value of V for which the slopes are the same. Therefore, there is no value of V for which the group velocity dispersion of the LP_{01} mode is equal to the group velocity dispersion of the LP_{11} mode.

Chapter 5 Selected Solutions

5.1 *Nonlinear terms*

(a) Expand the cube

$$\left(A_j \cos(\omega_j t) + A_k \cos(\omega_k t) + A_\ell \cos(\omega_\ell t)\right)^3$$

into a summation of ten product terms, one of which is

$$6 A_j A_k A_\ell \cos(\omega_j t) \cos(\omega_k t) \cos(\omega_\ell t).$$

Solution

The 10 terms of the expansion are

$$\begin{aligned}
6A_jA_kA_\ell \cos(\omega_j t)\cos(\omega_k t)\cos(\omega_\ell t) + 3A_j^2A_k\cos^2(\omega_j t)\cos(\omega_k t) \\
+3A_jA_k^2\cos(\omega_j t)\cos^2(\omega_k t) + 3A_j^2A_\ell\cos^2(\omega_j t)\cos(\omega_\ell t) \\
+3A_jA_\ell^2\cos(\omega_j t)\cos^2(\omega_\ell t) + A_j^3\cos^3(\omega_j t) + 3A_kA_\ell^2\cos(\omega_k t)\cos^2(\omega_\ell t) \\
+3A_k^2A_\ell\cos^2(\omega_k t)\cos(\omega_\ell t) + A_k^3\cos^3(\omega_k t) + A_\ell^3\cos^3(\omega_\ell t)\,.
\end{aligned}$$

The first term is the desired term.

(b) Using sum and difference cosine formulas, expand the product term $\cos(\omega_j t)\cos(\omega_k t)\cos(\omega_\ell t)$ and show that it can be written as

$$\begin{aligned}
\cos(\omega_j t)\cos(\omega_k t)\cos(\omega_\ell t) &= \tfrac{1}{4}\cos\big((\omega_j + \omega_k + \omega_\ell)\,t\big) \\
&+ \tfrac{1}{4}\cos\big((\omega_j - \omega_k + \omega_\ell)\,t\big) \\
&+ \tfrac{1}{4}\cos\big((\omega_j + \omega_k - \omega_\ell)\,t\big) \\
&+ \tfrac{1}{4}\cos\big((\omega_j - \omega_k - \omega_\ell)\,t\big)
\end{aligned}$$

Solution

Writing the cosines in terms of exponentials gives

$$\frac{1}{8}\left(e^{\mathrm{i}\omega_j t} + e^{-\mathrm{i}\omega_j t}\right)\left(e^{\mathrm{i}\omega_k t} + e^{-\mathrm{i}\omega_k t}\right)\left(e^{\mathrm{i}\omega_\ell t} + e^{-\mathrm{i}\omega_\ell t}\right).$$

Expanding and collecting the terms gives

$$\frac{1}{8}\Big[\left(e^{\mathrm{i}(\omega_j+\omega_k+\omega_\ell)t}+e^{-\mathrm{i}(\omega_j+\omega_k+\omega_\ell)t}\right)+\left(e^{\mathrm{i}(\omega_j-\omega_k+\omega_\ell)t}+e^{-\mathrm{i}(\omega_j-\omega_k+\omega_\ell)t}\right)$$
$$+\left(e^{\mathrm{i}(\omega_j+\omega_k-\omega_\ell)t}+e^{-\mathrm{i}(\omega_j+\omega_k-\omega_\ell)t}\right)+\left(e^{\mathrm{i}(\omega_j-\omega_k-\omega_\ell)t}+e^{-\mathrm{i}(\omega_j-\omega_k-\omega_\ell)t}\right)\Big]$$

Each of the four terms inside the square brackets is of the form $2\cos(\cdot)$ leading to the desired expression.

(c) What proportion of the total power on the left side is contained in the term $\cos\big((\omega_j-\omega_k-\omega_\ell)\,t\big)$ on the right side?

Solution
The term $\cos^2\big((\omega_j-\omega_k-\omega_\ell)\,t\big)$ contains $(1/4)^2 = 1/16$ of the total power. This can be verified by squaring both sides, expanding cosine squared as $\cos^2(\omega t) = (1+\cos 2\omega)/2$ and expanding the cosine in terms of exponentials.

5.2 *Effective Area*

(a) The commercial single-mode fiber known as Corning SMF-28 has a core diameter of $d = 8.3\mu$m. At an operating wavelength of 1550 nm, the fiber specifications are $\Delta = 0.0036$, $n \approx 1.47$ and $V \approx 2.09$. Using these values, determine the linearly polarized mode parameters p and q (cf. (3.3.27)).

Solution
Using an initial guess of $b = 0.4$, a root finding algorithm yields $b = 0.44$, $pa = 1.56$ and $qa = 1.39$.

(b) Calculate A_{eff} for the Corning SMF-28 fiber at $\lambda = 1.55\,\mu$m. Compare the calculated value with the measured value of 80 μm^2. (This requires numerical integration of (5.3.13).

Solution
The effective area is given by

$$\begin{aligned}
\mathcal{A}_{\text{eff}} &= 2\pi\frac{\left[\int_0^\infty |U(r)|^2 r\,\mathrm{d}r\right]^2}{\int_0^\infty |U(r)|^4 r\,\mathrm{d}r} \\
&= 2\pi a^2\frac{\left[\frac{1}{J_0^2(pa)}\int_0^1 J_0(paR)^2 R\,\mathrm{d}R+\frac{1}{K_0^2(qa)}\int_1^\infty K_0(qaR)^2 R\,\mathrm{d}R\right]^2}{\frac{1}{J_0^4(pa)}\int_0^1 J_0(paR)^4 R\,\mathrm{d}R+\frac{1}{K_0^4(qa)}\int_1^\infty K_0(qaR)^4 R\,\mathrm{d}R} \\
&= 2\pi(d/2)^2(0.703) = 76\,\text{microns}^2.
\end{aligned}$$

The calculated effective area is slightly less than the measured effective area.

5.5 *Phase matching*

(a) Starting with

$$a(z,t) \quad = \quad \sum_{m=-M}^{M} a_m(z,t) \exp\left(\mathrm{i}\left[m\Delta\omega\left(t - z/v_g\right) - (m\Delta\omega)^2 \beta_2 z/2\right]\right),$$

write out the terms for $M = 1$. For example, the term a_1 is

$$a_1 \quad = \quad a_1(z,t) \exp\left(\mathrm{i}\left[\Delta\omega\left(t - z/v_g\right) - \Delta\omega^2\beta_2 z/2\right]\right).$$

Solution

The other two terms for $M = 1$ are $a_0 = a_0(z,t)$ and $a_{-1} = \exp\left(\mathrm{i}\left[-\Delta\omega\left(t - z/v_g\right) - \Delta\omega^2\beta_2 z/2\right]\right)$.

(b) Form the product $(a^*_{-1} + a^*_0 + a^*_1)(a_{-1} + a_0 + a_1)^2$, then determine the phase-matched terms for which the frequencies sum to zero.

Solution

The total number of terms is given by

$$\begin{aligned}
&2a_0a_{-1}a_1^*(1)e^{-2\mathrm{i}\Delta\omega\tau} + 2a_0a_1a_{-1}^*e^{2\mathrm{i}\Delta\omega\tau} + a_{-1}^2a_{-1}^*e^{-\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} \\
&+a_{-1}^2a_0e^{-2\mathrm{i}\Delta\omega\tau - \mathrm{i}\beta_2\Delta\omega^2 z} + a_{-1}^2a_1^*e^{-3\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} + 2a_1a_{-1}a_{-1}^*e^{\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} \\
&+2a_0a_{-1}a_0e^{-\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} + 2a_1a_{-1}a_1^*e^{-\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} + a_0^2a_{-1}^*e^{\mathrm{i}\Delta\omega\tau + \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} \\
&+a_1^2a_{-1}^*e^{3\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} + a_1^2a_0^*e^{2\mathrm{i}\Delta\omega\tau - \mathrm{i}\beta_2\Delta\omega^2 z} + 2a_0a_1a_0^*e^{\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} \\
&+a_0^2a_1^*e^{\frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z - \mathrm{i}\Delta\omega\tau} + a_1^2a_1^*e^{\mathrm{i}\Delta\omega\tau - \frac{1}{2}\mathrm{i}\beta_2\Delta\omega^2 z} \\
&+2a_1a_{-1}a_0^*e^{-\mathrm{i}\beta_2\Delta\omega^2 z} + 2a_0a_{-1}a_{-1}^* + a_0^2a_0^* + 2a_0a_1a_1^*
\end{aligned}$$

The last four terms on the last line are the phase-matched terms because they do not have terms that include multiples of the temporal phase mismatch term $\Delta\omega\tau$.

(c) Collect the phase matched terms and show that

$$a^*a^2 \quad \approx \quad \left(|a_0|^2 + 2\,|a_1|^2 + 2\,|a_{-1}|^2\right) a_0 + 2a_1a_{-1}a_0^*e^{-\mathrm{i}\beta_2\Delta\omega^2 z},$$

as appears in (5.5.6).

Solution

Only the last four terms of the expansion are phase matched. Factoring out a_0 from the first three terms leads to the desired expression.

5.6 *Mean-squared output width for a weakly dispersive nonlinear fiber*

A *generalized gaussian pulse* is defined as

$$G(t) = \left(e^{-(t^2/2\sigma^2)^m}\right)^n,$$

where m and n are parameters. The area under this pulse can be expressed as and is

$$\int_{-\infty}^{\infty} G(t)\mathrm{d}t = 2^{3/2}\sigma\, n^{-1/2m}\,\Gamma\Big(1+(2m)^{-1}\Big), \tag{12}$$

where $\Gamma(x)$ is the gamma function defined as $\Gamma(k) \doteq \int_0^\infty x^{k-1}e^{-x}\mathrm{d}x$ (cf. (2.2.45)).

(a) Let $n = 1$. For pulse $G(t)$, determine the effective power

$$P_{\text{eff}}(m) = \frac{\int_{-\infty}^{\infty} |G(\tau)|^4\, \mathrm{d}\tau}{\int_{-\infty}^{\infty} |G(\tau)|^2\, \mathrm{d}\tau},$$

which was defined in (5.4.16). For $m = 1$, this term reduces to (5.4.19).

Solution

For $n = 1$, the pulse is given by

$$s(t) = e^{-(t^2/2\sigma^2)^m}.$$

Then

$$S_p(m) = \frac{\int_{-\infty}^{\infty} \left|e^{-(t^2/2\sigma^2)^m}\right|^4 \mathrm{d}\tau}{\int_{-\infty}^{\infty} \left|e^{-(t^2/2\sigma^2)^m}\right|^2 \mathrm{d}\tau} = \frac{\int_{-\infty}^{\infty} G(t,m,4)}{\int_{-\infty}^{\infty} G(t,m,2)},$$

where both the numerator and the denominator are in the form of $G(t)$ with $n = 4$ for the numerator and $n = 2$ for the denominator. Therefore

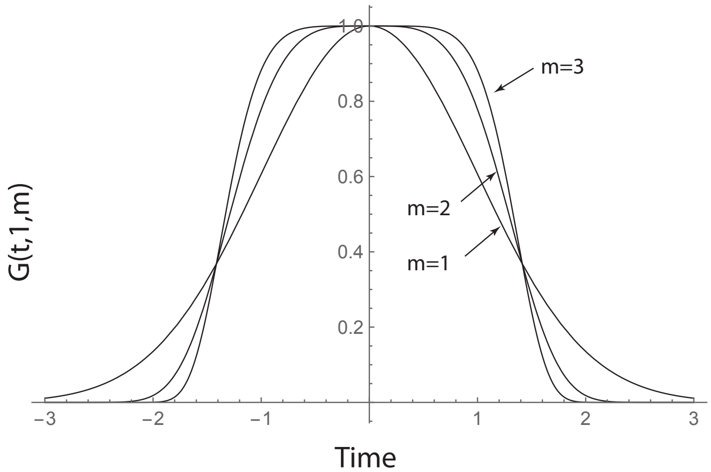
$$S_p(m) = \frac{2^{3/2}\sigma\, 4^{-1/2m}\,\Gamma\Big(1+(2m)^{-1}\Big)}{2^{3/2}\sigma\, 2^{-1/2m}\,\Gamma\Big(1+(2m)^{-1}\Big)} = 2^{-1/2m}.$$

For $m = 1$, the factor is $1/\sqrt{2}$ and agrees with (5.4.19).

(b) On the same figure, plot three generalized gaussian pulses for $1 \le m \le 3$ using $n = 1$ and $\sigma = 1$. Calculate the scaling factor $P_{\text{eff}}(m)$ for each pulse. On the basis of these three pulses, what kind of pulse experiences more pulse spreading in a weakly-dispersive fiber?

Why? (Note that the original problem statement used a different number of pulses.)

Solution
The plot is below. The effective scaling factors $P_{\text{eff}}(m)$ are $2^{-1/2}, 2^{-1/4}$ and $2^{-1/6}$ for $m = 1, 2$ and 3 respectively. The pulse for $m = 1$ has the least pulse spreading because it has the smallest bandwidth.



(c) Determine an expression for the instantaneous frequency shift $\omega(t) = \omega_c - \gamma \mathrm{d}P(z,t)/\mathrm{d}t$ given in (5.3.16) as a function of time in terms of the parameters m and σ for a pulse whose power is a generalized gaussian pulse with $n = 2$.

Solution
For $n = 2$, the power in a pulse is proportional to $|s(t)|^2$ so that

$$P(t) \quad \propto \quad e^{-(t^2/2\sigma^2)^{2m}}.$$

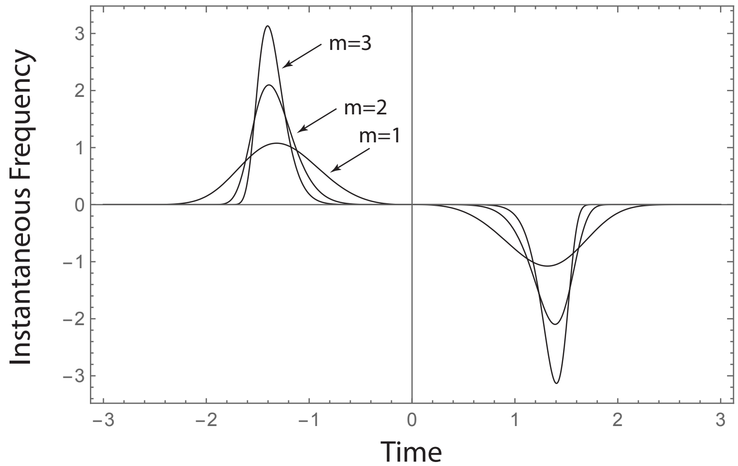
The instantaneous frequency is

$$\begin{aligned}\omega(t) &= \omega_c - \gamma\frac{\mathrm{d}P(z,t)}{\mathrm{d}t} \\ &= \omega_c - \sigma^{-1}\left(m\gamma 2^{2(1-m)} e^{-2^{-2m}(t/\sigma)^{4m}}\,(t/\sigma)^{4m-1}\right)\end{aligned}$$

(d) Plot $\omega(t)$ with $\sigma = 1$ for $1 \leq m \leq 3$. Compare these three curves with the results derived in part (b).

Solution
The plot is on the next page. The pulses for higher values of m have a larger instantaneous

frequency because the slope of the edges of the pulses shown in the previous figure is greater for larger m.



5.8 *Nonlinear fiber parameters*

A standard single mode fiber is given with D= 17 ps$/$(nm $\cdot$ km), $\gamma = 1.3$ radians$/$(W $\cdot$ km) and $\kappa = 0.2$ dB/km. A second fiber has D= 2.3 ps$/$(nm $\cdot$ km), $\gamma = 2$ radians$/$(W $\cdot$ km) and $\kappa = 0.2$ dB/km. An input gaussian pulse have a peak power of 50 mW and a root-mean squared temporal width of 50 ps.

(a) For each fiber, determine the nonlinear length $L_{\rm NL}$, the dispersion length L_D, the effective length $L_{\rm eff}$, and the walk-off length $L_{\rm wo}$ with subchannels separated by 100 GHz.

Solution

The table of the values is calculated below for a wavelength of 1.55 microns. Converting 0.2 dB/km into Nepers/km using 1 Neper/km = 4.34 dB/km gives $\kappa = 0.046\,\text{km}^{-1}$. The dispersion length uses $L_D = \sigma_{\rm in}^2/\beta_2$ with $\beta_2 = 125$ ps^2/km for D= 17 ps$/$(nm $\cdot$ km) and $\beta_2 = 18.4$ ps^2/km for D= 2.3 ps$/$(nm $\cdot$ km). For the walk-off length, use 100 GHz $\approx$ 0.8 nm at 1550 nm and equation (5.3.25).

	$L_{\rm NL}$ (km)	L_D (km)	$L_{\rm eff}$ (km)	$L_{\rm wo}$ (km)
Standard	15.4	18	21.7	3.7
Shifted	10	136	22	27

(b) For a single unamplified segment of fiber, derive the peak pulse power for each fiber such that the total accumulated phase shift is smaller than 0.1 radians.

Solution
The accumulated nonlinear phase $\phi_{\rm NL}$ is given in (5.3.20) and repeated here

$$\phi_{\rm NL} = \frac{L_{\rm eff}}{L_{\rm NL}} = \gamma L_{\rm eff} P_{\rm in}.$$

Setting $\phi_{\rm NL}$ to 0.1 radians and solving for the peak input power $P_{\rm in}$ gives

$$P_{\rm in} = \frac{0.1}{\gamma L_{\rm eff}}.$$

For the first fiber, $P_{\rm in}$ is about 3.5 mW. For the second fiber $P_{\rm in}$ is about 2.3 mW.

(c) For the same coupled power and the same pulse, which fiber produces a smaller nonlinear phase shift?

Solution
Standard fiber because γ is smaller.

(d) For the same coupled power and the same pulse, which fiber produces a larger dispersion-limited distance?

Solution
The dispersion-shifted fiber.

(e) Based on the results of the previous parts, discuss the circumstances under which each fiber should be preferred.

Solution
The dispersion-shifted fiber has a longer dispersion length, but a longer walk-off length and a shorter nonlinear length. Because the linear dispersion can be compensated while the nonlinear impairments are more difficult to compensate, low dispersion fiber tends not be used in high performance systems because it is less tolerant to nonlinear impairments.

Chapter 6 Selected Solutions

6.1 *Derivation of the Gordon distribution and its entropy*

(a) Starting with $p(\mathsf{m}) = Ku^{\mathsf{m}}$, derive

$$p(\mathsf{m}) \quad = \quad \frac{1}{1+\mathsf{S}}\left(\frac{\mathsf{S}}{1+\mathsf{S}}\right)^{\mathsf{m}},$$

satisfying the constraints $\sum_{n=0}^{\infty} \mathsf{m} f(\mathsf{m}) = \mathsf{S}$ and $\sum_{n=0}^{\infty} f(\mathsf{m}) = 1$.

Solution

Start with

$$p_{\underline{\mathsf{n}}}(\mathsf{n}) = Ku^{\mathsf{n}}$$

, where K and u are two constants that need to determined using the constraints $\sum_{n=0}^{\infty} p_{\underline{\mathsf{n}}}(\mathsf{n}) = 1$ and $\sum_{n=0}^{\infty} \mathsf{n} p_{\underline{\mathsf{n}}}(\mathsf{n}) = \mathsf{S}$. Substituting Ku^{n} into these two expressions and using the fact that $|u| < 1$ for a valid probability density function gives

$$\sum_{n=0}^{\infty} Ku^{\mathsf{n}} = \frac{K}{1-u} = 1$$

and

$$\sum_{n=0}^{\infty} K\mathsf{n}u^{\mathsf{n}} = \frac{Ku}{(1-u)^2} = \mathsf{S}.$$

Solving the first equation gives $u = 1 - K$. Substituting this expression into the second equation yields

$$K = \frac{1}{1+\mathsf{S}}$$

and

$$u = \frac{\mathsf{S}}{1+\mathsf{S}},$$

so that

$$p_{\underline{\mathsf{n}}}(\mathsf{n}) = Ku^{\mathsf{n}} = \frac{1}{1+\mathsf{S}}\left(\frac{\mathsf{S}}{1+\mathsf{S}}\right)^{\mathsf{n}}.$$

showing that the Gordon distribution is in the form of a geometric distribution.

(b) Using the form of $p(\mathsf{m})$ given in part (a), and the definition of the entropy H given by

$$H \quad = \quad -k \sum_{\mathsf{m}=0}^{\infty} p(\mathsf{m}) \log_e p(\mathsf{m}),$$

derive the entropy of a Gordon distribution, which is stated in Table 6.1.

Solution

The entropy is

$$\begin{aligned}
H &= -k\sum_{\mathsf{m}=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m}) \log_e p_{\underline{\mathsf{m}}}(\mathsf{m}) \\
&= -k\sum_{\mathsf{m}=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m}) \left(\log_e\left(\frac{1}{1+\mathsf{S}}\right) + \mathsf{m}\log_e\left(\frac{\mathsf{S}}{1+\mathsf{S}}\right)\right) \\
&= k\sum_{\mathsf{m}=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m}) \left(\log_e(1+\mathsf{S}) + \mathsf{m}\log_e\left(\frac{1+\mathsf{S}}{\mathsf{S}}\right)\right) \\
&= k\Bigg(\log_e(1+\mathsf{S}) \underbrace{\sum_{\mathsf{m}=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m})}_{=1} + \log_e\left(\frac{1+\mathsf{S}}{\mathsf{S}}\right) \underbrace{\sum_{\mathsf{m}=0}^{\infty} \mathsf{m} p_{\underline{\mathsf{m}}}(\mathsf{m})}_{=\mathsf{S}} \Bigg) \\
&= k\left(\log_e(1+\mathsf{S}) + \mathsf{S}\log_e\left(\frac{1+\mathsf{S}}{\mathsf{S}}\right)\right).
\end{aligned}$$

6.2 *Maximum-entropy distribution without a mean constraint*

Following the procedure used to derive the Gordon probability mass function given in (6.1.5), but removing the finite mean constraint on the probability distribution function, show that the maximum-entropy distribution on a finite number of states is the uniform probability density function given by

$$f(\mathsf{m}) \quad = \quad \frac{1}{\mathsf{M}},$$

where M is the number of states.

Solution

Start with

$$S = \sum_{m=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m}) \log_e p_{\underline{\mathsf{m}}}(\mathsf{m}) + C\sum_{m=0}^{\infty} p_{\underline{\mathsf{m}}}(\mathsf{m}),$$

which does not include the mean energy constraint. Following the same steps as used to derive (6.1.5) gives

$$\sum_{m=0}^{\infty} \left[\log_e p_{\underline{\mathsf{m}}}(\mathsf{m}) + 1 + C_2\right] f(\mathsf{m}) = 0.$$

Setting the term in brackets to zero and solving for $p_{\underline{\mathsf{m}}}(\mathsf{m})$ gives

$$p_{\underline{\mathsf{m}}}(\mathsf{m}) = e^{-(1+C)} = K,$$

which states that the value of the probability density function is a constant which is a uniform probability distribution. Therefore, when there is no mean energy constraint, every state is equally likely.

6.3 *The Bose-Einstein probability mass function and the Boltzmann probability density function*

(a) Starting with (6.1.9), derive an expression for the form of the probability density function of the energy $f(E)$, with $E = hf\mathsf{m}$.

Solution
Equation (6.1.9) is

$$\begin{aligned} p_{\underline{\mathsf{m}}}(\mathsf{m}) &= \;= \frac{1}{1+\mathsf{N}}\left(\frac{\mathsf{N}}{1+\mathsf{N}}\right)^{\mathsf{m}} \\ &= \left(1 - e^{-hf/kT_0}\right) e^{-\mathsf{m}(hf/kT_0)}. \end{aligned} \tag{13}$$

Using $f_{\underline{E}}(E)dE = f_{\underline{\mathsf{m}}}(\mathsf{m})d\mathsf{m}$ and $d\mathsf{m}/dE = 1/hf$ gives

$$f_{\underline{E}}(E) = \frac{1}{hf}\left(1 - e^{-hf/kT_0}\right) e^{-E/kT_0}.$$

(b) Is the resulting probability density function a valid continuous probability density function? Explain your answer.

Solution
The resulting function is not a valid continuous probability density function because it does not integrate to one when E is treated as a continuous variable.

(c) Take the limit of the expression in part (a) as hf goes to zero and show that the resulting expression is the Boltzmann probability density function.

Solution
Taking the limit as hf approaches zero gives

$$f_{\underline{E}}(E) = \frac{1}{kT_0} e^{-E/kT_0}.$$

This is a valid probability distribution and is the Boltzmann probability distribution.

6.5 *Coherence time and bandwidth*

The root-mean-squared coherence timewidth τ_{rms} of the autocorrelation function $R(\tau)$ (cf. (2.1.30)) has a different value than coherence timewidth τ_c defined by (2.2.59) and repeated here

$$\tau_c \doteq \frac{1}{|R(0)|^2}\int_{-\infty}^{\infty}|R(\tau)|^2\,\mathrm{d}\tau,$$

where the autocorrelation function $R(\tau)$ is the Fourier transform of the power density spectrum $\mathcal{S}(f)$ (cf. (2.2.55)). Determine the relationship between the root-mean-square coherence timewidth τ_{rms}, and the coherence timewidth τ_c for the following power density spectra in (a) and (b).

(a) $\mathcal{S}(f) = \frac{1}{\sqrt{2\pi}\sigma}e^{-f^2/2\sigma^2}$.

Solution
Using Parseval's relationship (cf. (2.1.18)) and the fact that $R(0)$ is the inverse Fourier transform of $\mathcal{S}(f)$ evaluated at $t = 0$ gives

$$\tau_c \doteq \frac{1}{|R(0)|^2}\int_{-\infty}^{\infty}|R(\tau)|^2\,\mathrm{d}\tau = \frac{\int_{-\infty}^{\infty}|\mathcal{S}(\tau)|^2\,\mathrm{d}\tau}{\left|\int_{-\infty}^{\infty}\mathcal{S}(f)\mathrm{d}f\right|^2} = \frac{1}{2\sigma\sqrt{\pi}}. \tag{14}$$

Using Table 2.1, the autocorrelation function $R(\tau)$ is

$$R(\tau) = \sqrt{2\pi}e^{-2\pi^2\sigma^2 t^2},$$

which, from inspection, has a root-mean squared timewidth τ_{rms} equal to

$$\tau_{\text{rms}} = \frac{1}{2\sigma\pi}.$$

The value τ_{rms} differs by a factor of $\sqrt{\pi}$ compared to the coherence timewidth τ_c given in (14).

(b) $\mathcal{S}(f) = e^{-|f|}$.

Solution
For the doubled sided-exponential, the coherence timewidth τ_c is

$$\tau_c \doteq \frac{1}{|R(0)|^2}\int_{-\infty}^{\infty}|R(\tau)|^2\,\mathrm{d}\tau = \frac{\int_{-\infty}^{\infty}e^{-2|f|}\mathrm{d}\tau}{\left(\int_{-\infty}^{\infty}e^{-|f|}\mathrm{d}f\right)^2} = \frac{1}{4}.$$

Using Table 2.1, the autocorrelation function $R(\tau)$ is in the form of a lorentzian function. The root-mean squared timewidth τ_{rms} of the this function is not defined because $x^2 f(x)$ does not go to zero as x goes to infinity.

(c) Citing a specific example, discuss why one definition of the coherence timewidth might be preferred over the other definition.

Solution
The example in part (b) shows that coherence timewidth τ_c can be defined whereas the root-mean squared timewidth τ_{rms} cannot be defined. Therefore, for a lorentzian function, the coherence time τ_c may be preferable to τ_{rms}.

6.7 *Filtered spontaneous emission*

The spontaneous emission noise in a single polarization of a lightwave is bandlimited using an ideal rectangular passband optical filter $h_o(t)$ with a complex-baseband transfer function given by

$$\begin{aligned} H_o(f) &= 1 \quad \text{for } |f| < B/2 \\ &\ 0 \qquad \text{otherwise.} \end{aligned}$$

The resulting filtered lightwave noise power has an expected value $\langle \underline{P} \rangle$. It is detected by an ideal photodetector with an impulse response $h(t)$ equal to $\delta(t)$.

(a) Determine the power density spectrum $\mathcal{S}_{\mathsf{g}}(f)$ of the arrival process $\mathsf{g}(t)$ within the photodetector.

Solution
The total noise density spectrum for the photoelectron generation rate process $g(t)$ within the photodetector is given by (6.4.15)

$$\mathcal{S}_{\mathsf{g}}(f) \quad = \quad \overline{\mathsf{R}} + \mathcal{S}_{\mathsf{R}}(f),$$

where the mean value for the photogeneration rate is given by $\overline{\mathsf{R}} = P\mathcal{R}/e$ The form for $\mathcal{S}_{\mathsf{R}}(f)$ is given in (6.4.16) where $\mathcal{S}_{P}(f)$ is the power density spectrum of the lightwave power given in (6.4.7) with the signal power P_s set equal to zero. This expression is scaled by a factor of $(\mathcal{R}/e)^2$ to express the signal in terms of photocounts. The normalized power spectrum of the bandlimited spontaneous emission is $\mathcal{S}_n(f) = 1/B$ for $|f| < B/2$. This is a rect function in frequency. Using (6.4.7) and noting that the convolution of a rect function with itself is a triangular function of the form $1 - 2\,|f|\,/B$, the power-density spectrum of the photoelectron generation rate process $g(t)$ within the photodetector includes both shot noise and intensity noise and can be written as

$$\mathcal{S}_{\mathsf{g}}(f) \;=\; \underbrace{\left(\frac{\mathcal{R}}{e}\right) P}_{\text{shot noise}} + \underbrace{\left(\frac{\mathcal{R}}{e}\right)^2 P^2\delta(f)}_{\text{mean}} + \underbrace{\left(\frac{\mathcal{R}}{e}\right)^2 P^2(1-2|f|/B)}_{\text{intensity fluctuations}}.$$

The form shows that the fluctuations for shot noise go as P, the fluctuations for intensity noise go as P^2.

(b) Determine the power density spectrum $\mathcal{S}(f)$ of the filtered electrical signal $r_{\mathsf{det}}(t)$ if the photodetected electrical signal is filtered by a detection filter with an impulse response $h(t) = e^{-t/\tau}u(t)$.

Solution

The transfer function $H(f)$ corresponding to $h(t)$ is

$$H(f) \;=\; \frac{\tau}{1 + \mathrm{i}2\pi\tau f}.$$

The power-density spectrum $\mathcal{S}_i(f)$ in units of A^2/Hz is given by $\mathcal{S}_i(f) = e^2\mathcal{S}_{\mathsf{g}}(f)$. Therefore, the power-density spectrum $\mathcal{S}(f)$ of the filtered directly-photodetected electrical signal into a unit resistance is given by $\mathcal{S}(f) = e^2\mathcal{S}_{\mathsf{g}}(f)H^2(f)$ where $\mathcal{S}_{\mathsf{g}}(f)$ was determined in part (a).

(c) Under what conditions are the statistics of the sample value $\underline{r}$ after the detection filter given by:

(i) exponential
(ii) gamma
(iii) gaussian

Solution

The three distributions correspond to systems for which the intensity noise is much larger than the shot noise so that continuous wave-optics distributions can be used.

(i) exponential

Solution

The statistics are exponential if there is only a single mode. This means that the coherence time of the optical source is on the order of the bandwidth of the filter function $H(f)$ so that $\tau_c \approx 1/B$.

(ii) gamma

Solution
The output distribution is, in general, a gamma distribution with the mean value given by P and the number of coherence intervals, which is approximately $B\tau_c$.

(iii) gaussian.

Solution
Gaussian output statistics are generated whenever the number of coherence intervals used to form the detection statistic becomes large.

6.8 *Bandlimited noise*

The electrical noise power generated by direct photodetection given in (6.5.3) was derived for $B\tau_c = 1$, where τ_c is the coherence timewidth defined in (2.2.59) and B is the passband noise-equivalent bandwidth (cf. (2.2.78)). The relationship between B and τ_c is valid when the lightwave-noise-suppressing filter is an ideal bandpass filter in the form of the rect function.

(a) Derive a corresponding expression for $B\tau_c$ for a lightwave noise-suppressing filter defined by a gaussian function with a root-mean-squared width σ equal to B.

Solution
Let the gaussian function in the frequency domain be given as

$$H(f) \doteq e^{-f^2/2\sigma^2}.$$

We want to determine the product $B\tau_c = B/B_c$ where B is equal to the passband noise-equivalent bandwidth B_N given in (2.2.77) and $B_c = 1/\tau_c$ is the power equivalent width. Using

$$B_N \doteq \frac{1}{G}\int_{-\infty}^{\infty} |H(f)|^2 \,\mathrm{d}f.$$

With $G = 1$ and $H(f)$ given above, this gives

$$B = B_N \doteq \int_{-\infty}^{\infty} \left| e^{-f^2/2\sigma^2} \right|^2 \mathrm{d}f = \sigma\sqrt{\pi}.$$

The filtered noise power density spectrum is given by $\mathcal{S}_n(f) = N_0|H(f)|^2$ with $|H(f)|^2 = \left|e^{-f^2/2\sigma^2}\right|^2$. Using (2.2.78) for B_c gives

$$\begin{aligned} B_c &= \frac{\left(\int_{-\infty}^{\infty} \mathcal{S}_n(f)\mathrm{d}f\right)^2}{\int_{-\infty}^{\infty} \mathcal{S}_n^2(f)\mathrm{d}f} \\ &= \frac{(\int_{-\infty}^{\infty} \left|e^{-f^2/2\sigma^2}\right|^2 \mathrm{d}f)^2}{\int_{-\infty}^{\infty} \left|e^{-f^2/2\sigma^2}\right|^4 \mathrm{d}f} \quad = \quad \frac{\pi\sigma^2}{\sigma\sqrt{\pi/2}} \quad = \quad \sigma\sqrt{\pi/2}. \end{aligned}$$

The ratio is

$$\frac{B}{B_c} \quad = \quad \frac{\sigma\sqrt{\pi}}{\sigma\sqrt{\pi/2}} \quad = \quad \sqrt{2}.$$

Therefore in constrast to the ideal retangular noise-suppressing filter, the noise equivalent bandwidth B_N is not equal to the power-equivalent bandwidth $B_c = 1/\tau_c$.

(b) Quantitatively explain how the value of $B\tau_c$ affects the statistics of the sample determined over an interval of duration T.

Solution
The value of $B\tau_c$ is an estimate of the number of temporal degrees of freedom in the system. This value dictates the ability to convey information in time.

6.9 *Degrees of freedom of lorentzian-filtered noise*

Let the autocorrelation function of the noise process $n(t)$ be given by

$$R_n(\tau) \quad = \quad Ne^{-\alpha|\tau|}.$$

(a) Show that this autocorrelation function is generated by filtering white noise with a filter that has the transfer function

$$H(\omega) \quad = \quad N\frac{2\alpha}{\alpha^2 + \omega^2}.$$

A filter of this form is called a lorentzian filter.

Solution
This expression can be derived using Table 2.1, and the scaling property of the Fourier transform using an angular frequency ω.

(b) Write the integral in (6.6.4) using symmetric limits. Separate that integral into two regions and differentiate twice to produce a second-order differential equation of the form

$$\frac{\mathrm{d}^2\psi_k(t_1)}{\mathrm{d}t_1^2} + b_k^2\psi_k(t_1) \quad = \quad 0.$$

Determine the expression for b_k in terms of λ_k, α, and N.

Solution

Writing the integral equation to solve in symmetric form gives

$$\lambda_k\psi_k(t_1) = N\int_{-T}^{T} e^{-\alpha|t_1-t_2|}\psi_k(t_2)\mathrm{d}t_2. \tag{15}$$

Separate this integral into two regions and differentiate twice.

$$\lambda_k\psi_k(t_1) = N\int_{-T}^{t_1} e^{-\alpha(t_1-t_2)}\psi_k(t_2)\mathrm{d}t_2 + N\int_{t_1}^{T} e^{-\alpha(t_2-t_1)}\psi_k(t_2)\mathrm{d}t_2. \tag{16}$$

The first derivative is

$$\lambda_k\frac{\mathrm{d}\psi_k(t_1)}{\mathrm{d}t_1} = -N\alpha e^{-\alpha t_1}\int_{-T}^{t_1} e^{\alpha t_2}\psi_k(t_2)\mathrm{d}t_2 + N\alpha e^{\alpha t_1}\int_{t_1}^{T} e^{-\alpha t_2}\psi_k(t_2)\mathrm{d}t_2. \tag{17}$$

Combining the two expressions, the second derivative can be written as

$$\lambda_k\frac{\mathrm{d}^2\psi_k(t_1)}{\mathrm{d}t_1^2} \quad = \quad N\alpha^2\int_{-T}^{T} e^{-\alpha|t_1-t_2|}\psi_k(t_2)\mathrm{d}t_2 - 2N\alpha\psi_k(t_1)$$

Noting that the first term on the right is $\alpha^2\lambda_k\psi_k(t_1)$ (cf. (15)), we can write

$$\frac{\mathrm{d}^2\psi_k(t_1)}{\mathrm{d}t_1^2} = \left(\frac{\alpha^2(\lambda_k - 2N/\alpha)}{\lambda_k}\right)\psi_k(t_1).$$

Rewrite this equation as

$$\frac{\mathrm{d}^2\psi_k(t_1)}{\mathrm{d}t_1^2} + b^2\psi_k(t_1) = 0 \tag{18}$$

where

$$\begin{aligned} b^2 &= -\frac{\alpha^2(\lambda_k - 2N/\alpha)}{\lambda_k} \\ &= \frac{2N\alpha}{\lambda_k} - \alpha^2. \end{aligned}$$

Solving for the eigenvalue λ_k in terms of b gives

$$\lambda_k = \frac{2N\alpha}{\alpha^2 + b^2} = \frac{2N\alpha}{(a + \mathrm{i}b)(a - \mathrm{i}b)}. \tag{19}$$

(c) Now assume a solution of the form of

$$\psi_k(t_1) \quad = \quad c_1 e^{\mathrm{i}bt} + c_2 e^{-\mathrm{i}bt}.$$

Substitute this form into the original integral equation and perform the integration for each of the two regions.

Solution

Suppose that

$$0 < \lambda_k < \frac{2N}{\alpha},$$

so that b^2 is real with $0 < b^2 < \infty$. The solution to (18) is then be written in the following form

$$\psi_k(t_1) = c_1 e^{\mathrm{i}bt} + c_2 e^{-\mathrm{i}bt}. \tag{20}$$

Substitute this general solution into (16) and perform the integration

$$\begin{aligned}
\lambda_k \left(c_1 e^{\mathrm{i}bt_1} + c_2 e^{-\mathrm{i}bt_1}\right) \quad &= \quad N e^{-\alpha t_1} \int_{-T}^{t_1} \left(c_1 e^{(\alpha+\mathrm{i}b)t_2} + c_2 e^{-(-\alpha+\mathrm{i}b)t_2}\right) \mathrm{d}t_2 \\
&\quad + N e^{\alpha t_1} \int_{t_1}^{T} \left(c_1 e^{(-\alpha+\mathrm{i}b)t_2} + c_2 e^{-(\alpha+\mathrm{i}b)t_2}\right) \mathrm{d}t_2
\end{aligned}$$

Evaluating the integrals and collecting terms gives

$$\begin{aligned}
\frac{2N\alpha}{(\alpha+\mathrm{i}b)(\alpha-\mathrm{i}b)} \left(c_1 e^{\mathrm{i}bt_1} + c_2 e^{-\mathrm{i}bt_1}\right) \quad &= \quad N c_1 e^{\mathrm{i}bt_1} \underbrace{\left(e^{-\alpha t} \frac{e^{-(\alpha+\mathrm{i}b)T}}{\alpha+\mathrm{i}b} - e^{\alpha t} \frac{e^{-(\alpha-\mathrm{i}b)T}}{-\alpha+\mathrm{i}b}\right)}_{A} \\
&\quad + \quad N c_2 e^{-\mathrm{i}bt_2} \underbrace{\left(-e^{-\alpha t} \frac{e^{-(\alpha-\mathrm{i}b)T}}{\alpha-\mathrm{i}b} + e^{\alpha t} \frac{e^{-(\alpha+\mathrm{i}b)T}}{-\alpha-\mathrm{i}b}\right)}_{B}.
\end{aligned}$$

In order for the equation to be satisfied the two terms labeled A and B must be equal to each other and equal to the term on the left side of the equation. Equating these terms and reordering gives

$$e^{-\alpha t_1} \left(\frac{c_1 e^{-(\alpha+\mathrm{i}b)T}}{\alpha+\mathrm{i}b} + \frac{c_2 e^{-(\alpha-\mathrm{i}b)T}}{\alpha-\mathrm{i}b}\right) - e^{\alpha t_1} \left(\frac{c_1 e^{-(\alpha-\mathrm{i}b)T}}{-\alpha+\mathrm{i}b} + \frac{c_2 e^{-(\alpha+\mathrm{i}b)T}}{-\alpha-\mathrm{i}b}\right) = 0.$$

(d) Show that the resulting expression can be satisfied for all time only if $c_1 = c_2$ or if $c_1 = -c_2$.

Solution

This condition can be directly verified.

(e) Setting $t_1 = T$, derive the expression that must be satisfied if $c_1 = c_2$.

Solution

When $c_1 = c_2$ and $t_1 = T$, the following equation must be satisfied

$$\frac{4e^{-\alpha T}\sinh(\alpha T)(b\sin(bT) - \alpha\cos(bT))}{\alpha^2 + b^2} = 0,$$

which means that

$$b\sin(bT) - \alpha\cos(bT) = 0$$

or

$$\tan(bT) = \frac{\alpha}{b}.$$

(f) Setting $t_1 = T$, derive the expression that must be satisfied if $c_1 = -c_2$.

Solution

When $c_1 = -c_2$, the equation is

$$-\frac{4ie^{\alpha(-T)}\cosh(\alpha T)(\alpha\sin(bT) + b\cos(bT))}{\alpha^2 + b^2} = 0$$

or

$$\tan(bT) = -\frac{b}{\alpha}.$$

(g) A solution to either of the two previous equations will produce an eigenvalue. By combining these two equations, show that

$$\left(\tan(b_k T) + \frac{b_k T}{\alpha T}\right)\left(\tan(b_k T) - \frac{\alpha T}{b_k T}\right) \quad = \quad 0.$$

Solution

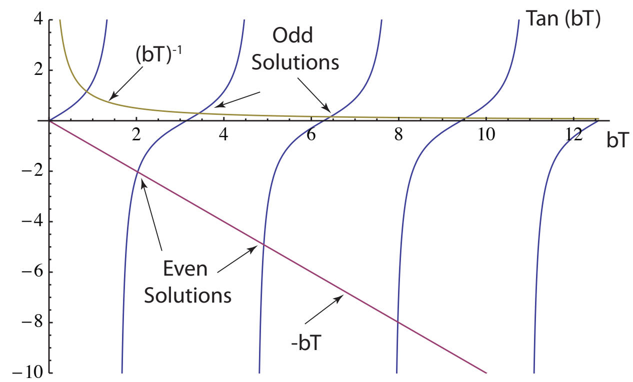
Combining these equations and multiplying the top and bottom by T gives

$$\left(\tan(bT) + \frac{bT}{\alpha T}\right)\left(\tan(bT) - \frac{\alpha T}{bT}\right) = 0.$$

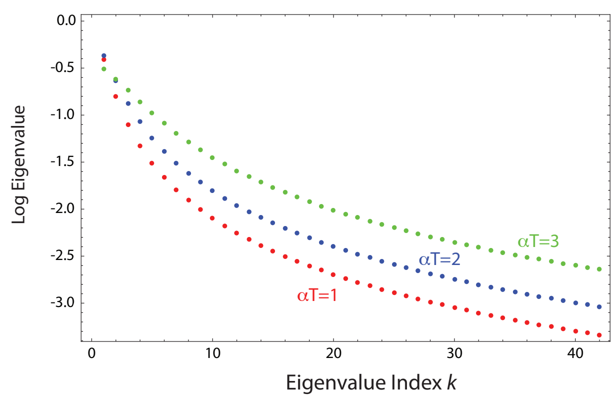
(h) Using the relationship between b_k and λ_k derived in part (b), plot the eigenvalues λ_k on a log plot. Compare the distribution of the eigenvalues for lorentzian-filtered noise for $\alpha T = 5$ to the eigenvalues for an ideal bandpass filter for $TB = 5$ given in Figure 6.8.

Solution

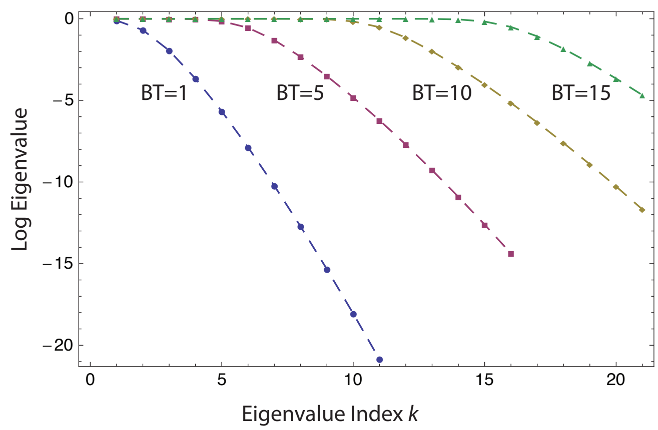
The solutions to this equation can be determined graphically as is shown in the figure for the value of $\alpha T = 1$.



Once the values b_k are determined, the corresponding eigenvalues are given by (19) with $b \to b_k$. A plot of the eigenvalues for several values of αT are shown in the figure below.



Now compare the eigenvalues for this plot to the plot of the eigenvalues for the ideal rectangular filter given in Figure 6.8 and repeated here.



It can be seen that the eigenvalues of the lorentzian-filtered noise roll off much more slowly compared to the eigenvalues for a rectangular filter.

(i)) Comment on the distribution of the eigenvalues for both kinds of filters with respect to the distribution of the entropy, which defines the ability of each degree of freedom to convey information.

Solution

The distribution of entropy per degree of freedom tracks the distribution of the eigenvalues. For an ideal rectangular passband filter, the distribution of the eigenvalues (or entropy) is flat up to TB and then rapidly rolls off. Therefore, for this kind of filter, the number of degrees of freedom is well-approximated by TB with the entropy for each degree of freedom being nearly the same because the distribution of eigenvalues is nearly the same. This is not the case for the lorentzian filter. For this case, every degree of freedom has a different entropy.

6.10 *Sum of Poisson random variables*

Prove that if the sum of two random variables $\underline{\mathsf{m}_3} = \underline{\mathsf{m}_1} + \underline{\mathsf{m}_2}$, is Poisson and either of the two summands, $\underline{\mathsf{m}_1}$ or $\underline{\mathsf{m}_2}$, is Poisson, then the other summand is Poisson as well.

Solution

Let $p_1(\mathsf{m})$, and $p_3(\mathsf{m})$ be two Poisson probability distributions with mean values E_1 and E_3 respectively. Then the probability distribution $p_3(\mathsf{m})$ for $\underline{\mathsf{m}}_3$ is the convolution $p_3(\mathsf{m}) = p_1(\mathsf{m}) \circledast p_2(\mathsf{m})$. The convolution property of a Fourier transform states that the two characteristic functions satisfy

$$C_3(\omega) \quad = \quad C_1(\omega)C_2(\omega),$$

so that

$$C_2(\omega) = \frac{C_3(\omega)}{C_1(\omega)}.$$

Substituting $C_1(\omega) = e^{\mathsf{E}_1\left(e^{i\omega}-1\right)}$ and $C_3(\omega) = e^{\mathsf{E}_3\left(e^{i\omega}-1\right)}$ gives

$$C_2(\omega) = \frac{e^{\mathsf{E}_3\left(e^{i\omega}-1\right)}}{e^{\mathsf{E}_1\left(e^{i\omega}-1\right)}} = e^{(\mathsf{E}_3-\mathsf{E}_1)(e^{i\omega}-1)}$$

which is the characteristic function of a Poisson random variable with mean $\mathsf{E}_3 - \mathsf{E}_1$.

6.11 *Circular symmetry*

A product bivariate random variable with bivariate probability density function $f(x,y) = g(x)g(y)$ is known to be circularly symmetric in the (x,y) coordinate system. Does this mean that it is a bivariate gaussian random variable?

Solution

Working with the squared magnitude instead of the amplitude, the probability density function for the squared magnitude of circularly symmetric function can be written as

$$f(x,y) = Ah\left(x^2+y^2\right),$$

where A normalizes the distribution. If this function is separable, then

$$Ah\left(x^2+y^2\right) = B^2 g(x)g(y)$$

where B normalizes the one-dimensional probability density functions $g(\cdot)$. This equation can only be satisfied when both $g(\cdot)$ and $h(\cdot)$ are exponential functions because the exponential function as the unique property that the product of two exponential terms is a single exponential with an exponent that is the sum of the arguments the separate terms. This property is the inverse of the property of logarithms that the logarithm of a product is the sum of the logarithms of the terms in the product. Therefore, the squared magnitude must be an exponential function. Applying the constraint of circular symmetry means that the exponential function for the squared magnitude can be written as a product distribution with the joint probability density in polar coordinates given by (6.2.10)). Transforming from polar coordinates to cartesian coordinates recovers the bivariate gaussian distribution given in (6.2.8).

6.12 *Derivation of the negative binomial probability mass function*

Using the integral

$$\int_0^\infty e^{-\mathsf{E}(1+\mathsf{N_{sp}}^{-1})}\mathsf{E}^{(K-1+\mathsf{m})}d\mathsf{E} \quad = \quad \left(1+\mathsf{N_{sp}}^{-1}\right)^{-(K+\mathsf{m})}\Gamma(K+\mathsf{m}),$$

show that the Poisson transform of a gamma probability density function is equal to the negative binomial probability mass function.

Solution

The integral is

$$p_{\underline{\mathsf{m}}}(\mathsf{m}) \quad = \quad \frac{1}{\Gamma(K)}\frac{1}{\langle\underline{\mathsf{m}}\rangle\mathsf{m}!}\int_0^\infty \mathsf{E}^{\mathsf{m}}\left(\mathsf{E}/\langle\underline{\mathsf{m}}\rangle\right)^{K-1}e^{-\mathsf{E}(1+\langle\underline{\mathsf{m}}\rangle^{-1})}d\mathsf{E}.$$

The integral may be written as

$$p_{\underline{\mathsf{m}}}(\mathsf{m}) \quad = \quad \frac{1}{\Gamma(K)}\frac{1}{\langle\underline{\mathsf{m}}\rangle^{K}\mathsf{m}!}\int_0^\infty \mathsf{E}^{(\mathsf{m}+K-1)}e^{-\mathsf{E}(1+\langle\underline{\mathsf{m}}\rangle^{-1})}d\mathsf{E}.$$

Using the integral given in the problem statement yields

$$p_{\underline{\mathsf{m}}}(\mathsf{m}) \quad = \quad \frac{1}{\Gamma(K)}\frac{1}{\langle\underline{\mathsf{m}}\rangle^{K}\mathsf{m}!}\left(1+\langle\underline{\mathsf{m}}\rangle^{-1}\right)^{-(K+\mathsf{m})}\Gamma(K+\mathsf{m})$$

Collecting terms and using $\mathsf{m}! = \Gamma(\mathsf{m}+1)$ gives

$$\frac{\Gamma(K+\mathsf{m})}{\Gamma(K)\Gamma(\mathsf{m}+1)}\frac{1}{\langle\underline{\mathsf{m}}\rangle^{K}}\left(\frac{\langle\underline{\mathsf{m}}\rangle}{1+\langle\underline{\mathsf{m}}\rangle}\right)^{K+\mathsf{m}}.$$

Separating the last term into a term for K and a term for m and using $\frac{\Gamma(K+\mathsf{m})}{\Gamma(K)\Gamma(\mathsf{m}+1)} = \underbrace{\begin{pmatrix} K-1+\mathsf{m} \\ \mathsf{m} \end{pmatrix}}$ we have

$$\begin{pmatrix} K-1+\mathsf{m} \\ \mathsf{m} \end{pmatrix}\left(\frac{1}{1+\langle\underline{\mathsf{m}}\rangle}\right)^{K}\left(\frac{\langle\underline{\mathsf{m}}\rangle}{1+\langle\underline{\mathsf{m}}\rangle}\right)^{\mathsf{m}}$$

which is (6.5.11).

6.13 *Derivation of the mean and the variance of the number of counts*

Starting with

$$\begin{aligned} C_{\mathsf{m}}(\omega) &= \int_0^\infty e^{\mathsf{E}\left(e^{\mathrm{i}\omega}-1\right)}f(\mathsf{E})d\mathsf{E} \\ &= \left\langle e^{\mathsf{E}\left(e^{\mathrm{i}\omega}-1\right)}\right\rangle \\ &= C_{\mathsf{E}}\left(-\mathrm{i}\left(e^{\mathrm{i}\omega}-1\right)\right), \end{aligned}$$

as given in (6.3.7), derive the mean and variance of $p(\mathsf{m})$ in terms of $\langle \underline{\mathsf{E}} \rangle$ and σ^2_{E}. The probability mass function $p(\mathsf{m})$ is the Poisson transform of the probability density function $f(\mathsf{E})$ for the mean number of counts. The result should agree with the terms in (6.3.8).

Solution

The characteristic function is given in (6.3.7)

$$C_{\underline{\mathsf{m}}}(\omega) = \left\langle \exp\left(\underline{\mathsf{E}} e^{\mathrm{i}\omega} - 1\right) \right\rangle$$

The moments are determined using (2.2.17)

$$\langle \underline{x}^n \rangle = \left| \frac{1}{\mathrm{i}^n} \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} C_{\underline{x}}(\omega) \right|_{\omega=0}$$

The mean value is then

$$\begin{aligned}
\langle \underline{\mathsf{m}} \rangle &= \left| \frac{1}{\mathrm{i}^n} \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} C_{\underline{\mathsf{m}}}(\omega) \right|_{\omega=0} \\
&= \left\langle \left| \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\omega} \exp\left(\underline{\mathsf{E}} e^{\mathrm{i}\omega} - 1\right) \right|_{\omega=0} \right\rangle \\
&= \left\langle \left| \frac{1}{\mathrm{i}} (\mathrm{i}\underline{\mathsf{E}}) e^{\mathrm{i}\omega} \exp\left(\underline{\mathsf{E}}\left(e^{\mathrm{i}\omega} - 1\right)\right) \right|_{\omega=0} \right\rangle \\
&= \langle \underline{\mathsf{E}} \rangle
\end{aligned}$$

where the order of the expectation and differentiation has been interchanged. This expression is (6.3.8a). The mean square-value is

$$\begin{aligned}
\langle \underline{\mathsf{m}}^2 \rangle &= \left\langle \left| \frac{1}{\mathrm{i}^2} \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} \exp\left(\underline{\mathsf{E}} e^{\mathrm{i}\omega} - 1\right) \right|_{\omega=0} \right\rangle \\
&= \left\langle e^{\underline{\mathsf{E}}\left(e^{\mathrm{i}\omega}-1\right)} \left[\underline{\mathsf{E}} e^{\mathrm{i}\omega} + \left(\underline{\mathsf{E}} e^{\mathrm{i}\omega}\right)^2 \right] \Big|_{\omega=0} \right\rangle \\
&= \langle \underline{\mathsf{E}} \rangle + \langle \underline{\mathsf{E}}^2 \rangle
\end{aligned}$$

The variance is then

$$\begin{aligned}
\sigma^2_{\mathsf{m}} &= \langle \underline{\mathsf{m}}^2 \rangle - \langle \underline{\mathsf{m}} \rangle^2 \\
&= \langle \underline{\mathsf{E}} \rangle + \langle \underline{\mathsf{E}}^2 \rangle - \langle \underline{\mathsf{E}} \rangle^2 \\
&= \langle \underline{\mathsf{E}} \rangle + \sigma^2_{\underline{\mathsf{E}}},
\end{aligned}$$

which is (6.3.8b).

6.15 *Filtered shot noise and thermal noise*

An electrical waveform $r(t)$ is generated by the direct photodetection of a random lightwave signal with a nonstationary arrival rate given by $\mathsf{R}(t) = e^{-t/T}u(t)$ where $u(t)$ is the unit-step function. The photodetector has an impulse response given by $h(t) = e^{-t/4T}u(t)$. Using Campbell's theorem, derive the mean and the variance of the output electrical waveform $r(t)$.

Solution

The mean and the variance are given by Campbell's theorem (6.7.2)

$$p(T) = \langle \mathsf{R}(t)\rangle \circledast h(t)\big|_{t=T}$$
$$\sigma^2(T) = \langle \mathsf{R}(t)\rangle \circledast h^2(t)\big|_{t=T}$$

The mean is

$$\begin{aligned} p(T) &= e^{-t/T}u(t) \circledast e^{-t/4T}u(t) \\ &= \int_0^t e^{-\tau/4T} e^{-(t-\tau)/T} d\tau \bigg|_{t=T} \\ &= \frac{4\left(e^{3/4}-1\right)T}{3e}. \end{aligned}$$

The variance is

$$\begin{aligned} \sigma^2(T) &= e^{-t/T}u(t) \circledast e^{-t/2T}u(t) \\ &= \int_0^t e^{-\tau/2T} e^{-(t-\tau)/T} d\tau \bigg|_{t=T} \\ &= \frac{2\left(\sqrt{e}-1\right)T}{e}. \end{aligned}$$

6.16 *Isserlis theorem*

Isserlis theorem states that the expectations of four jointly gaussian random variables, $\underline{X}_1$, $\underline{X}_2$, $\underline{X}_3$, and $\underline{X}_4$, satisfy

$$\langle \underline{X}_1\underline{X}_2\underline{X}_3\underline{X}_4\rangle = \langle \underline{X}_1\underline{X}_2\rangle\langle \underline{X}_3\underline{X}_4\rangle + \langle \underline{X}_1\underline{X}_3\rangle\langle \underline{X}_2\underline{X}_4\rangle + \langle \underline{X}_1\underline{X}_4\rangle\langle \underline{X}_2\underline{X}_3\rangle.$$

Using the asserted Isserlis theorem, prove (6.4.2) for circularly-symmetric gaussian random variables.

Solution

The power autocorrelation function is

$$\begin{aligned} R_{P_n}(\tau) &= \langle P_n(t)P_n(t+\tau)\rangle \\ &= \langle n(t)n^*(t)n(t+\tau)n^*(t+\tau\rangle \end{aligned}$$

Now set $\underline{X}_1 = n(t)$, $\underline{X}_2 = n^*(t)$, $\underline{X}_3 = n(t+\tau)$, and $\underline{X}_4 = n^*(t+\tau)$. This gives

$$\begin{aligned} \frac{1}{4}\langle n(t)n^*(t)n(t+\tau)n^*(t+\tau\rangle &= \frac{1}{4}\underbrace{\langle n(t)n^*(t)\rangle}_{2P_n}\underbrace{\langle n(t+\tau)n^*(t+\tau\rangle}_{2P_n} + \\ &\quad \frac{1}{4}\underbrace{\langle n(t)n^*(t+\tau)\rangle}_{2R_n}\underbrace{\langle n^*(t)n(t+\tau)\rangle}_{2R_n} + \\ &\quad \frac{1}{4}\underbrace{\langle n(t+\tau)n(t)\rangle}_{0}\underbrace{\langle n^*(t+\tau)n^*(t)\rangle}_{0} \\ &= P_n + R_n, \end{aligned}$$

where the last two terms are pseudocovariance functions of circularly-symmetric gaussian random variables and so are zero (cf. Section 2.2.1).

Chapter 7 Selected Solutions

7.1 *Three-dB coupler*

The governing equations for a symmetric directional coupler with inputs $s_1(t,z)$ and $s_2(t,z)$ are

$$\begin{aligned}\frac{\mathrm{d}s_1(t,z)}{\mathrm{d}z} &= -\mathrm{i}\kappa s_2(t,z)\\ \frac{\mathrm{d}s_2(t,z)}{\mathrm{d}z} &= -\mathrm{i}\kappa s_1(t,z),\end{aligned}$$

where κ is the coupling coefficient between the modes in each waveguide, and each mode has a z dependence given by $e^{-\mathrm{i}\beta z}$. The output signals are defined as $z_1(t) \doteq s_1(t,L)$ and $z_2(t) \doteq s_2(t,L)$.

(a) Let the two inputs to the two paths of the directional coupler be $s_1(t,0) = s$ and $s_2(t,0) = 0$. Solve for $z_1(t)$ and $z_2(t)$ and determine the length L such that the two output signals are in phase quadrature.

Solution

Taking the derivative of the first equation, substituting the second equation on the right side and solving for $s_1(t,z)$ using the boundary condition $s_1(t,0) = s$ yields

$$s_1(z) = s\cos(\kappa z).$$

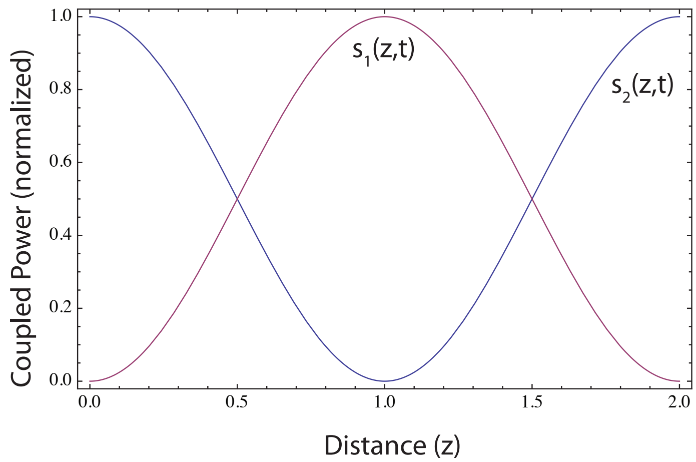
Taking the derivative of this equation yields

$$s_2(z) = -s\mathrm{i}\sin(\kappa z).$$

(b) Plot the power in each mode as a function of L and determine the minimum value of L that produces a 3-dB coupler.

Solution

The power for each waveguide is $s^2\cos^2(\kappa z)$ and $s^2\sin^2(\kappa z)$. The plot is on the next page using $\kappa = \pi/2$.



The value for κL that produces a 3-dB coupler is $1/\sqrt{2}$.

(c) Determine the minimum value of L that produces a power splitter with 10% of the lightwave power coupled into one path and 90% of the lightwave power coupled into the other path.

Solution
The value for κL that produces a 10% coupler is $\cos^{-1}(1/\sqrt{10})$.

(d) let the input to one path be $s_1(t, 0) = A$, and let the input to the other path be $s_2(t, 0) = B$. Show that for a proper choice of L, the output signals can be expressed as

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix},$$

which is the relationship for a 180-degree hybrid coupler given in (7.1.2).

Solution
For inputs A and B, the coupling matrix can be written as

$$\begin{bmatrix} s_1(t, L) \\ s_2(t, L) \end{bmatrix} = \begin{bmatrix} \cos \kappa t & -\mathrm{i} \sin \kappa t \\ -\mathrm{i} \sin \kappa t & \cos \kappa t \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Setting $\kappa L = \pi/4$ yields the desired result.

7.2 *Lossless couplers*

For a coupler to be lossless, the output power in the two output waveguides must equal the input power in the two waveguides so that

$$P_{\text{in}_1} + P_{\text{in}_2} \quad = \quad P_{\text{out}_2} + P_{\text{out}_2}$$

where $P = |s|^2$ is the root-mean-squared lightwave power and s is the complex lightwave amplitude. Let

$$\mathbf{s} \quad = \quad \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix},$$

be the vector of the two signals defined at either the input or the output of the coupler.
(a) Show that when the coupler is lossless,

$$\mathbf{s}_{\text{in}}^{\dagger}\mathbf{s}_{\text{in}} \quad = \quad \mathbf{s}_{\text{out}}^{\dagger}\mathbf{s}_{\text{out}},$$

where † denotes the conjugate transpose and $\mathbf{s}_{\text{out}} = \mathbb{T}\mathbf{s}_{\text{in}}$.

Solution
We must show that $P_{\text{in}_1} + P_{\text{in}_2} = P_{\text{in}_2} + P_{\text{in}_2}$. Forming the matrix product we have

$$[s_{\text{in}_1}^* \ s_{\text{in}_2}^*] \begin{bmatrix} s_{\text{in}_1} \\ s_{\text{in}_2} \end{bmatrix} \quad = \quad |s_{\text{in}_1}^*|^2 + |s_{\text{in}_2}^*|^2 \ = \ P_{\text{in}_1} + P_{\text{in}_2}.$$

The same result holds for the output.

(b) Show that $\mathbb{T} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ does not satisfy the condition derived in part (a). This means that combining two spatially-distinct input modes at the same carrier frequency into a single output mode cannot be implemented by a lossless transformation.
Solution
If $\mathbf{s}_{\text{out}} = \mathbb{T}\mathbf{s}_{\text{in}}$ then $\mathbf{s}_{\text{out}}{}^{\dagger} = (\mathbb{T}\mathbf{s}_{\text{in}})^{\dagger} = \mathbf{s}_{\text{in}}^{\dagger}\mathbb{T}^{\dagger}$. Form the product

$$P_{\text{out}} \quad = \quad \mathbf{s}^{\dagger}{}_{\text{out}}\mathbf{s}_{\text{out}} = \mathbf{s}_{\text{in}}^{\dagger}\mathbb{T}^{\dagger}\mathbb{T}\mathbf{s}_{\text{in}}.$$

In order for the power to be conserved, the product $\mathbb{T}^{\dagger}\mathbb{T}$ must be an identity matrix or equivalently, $\mathbb{T}$ must be a unitary matrix that satisfies $\mathbb{T}^{-1} = \mathbb{T}^{\dagger}$. When

$$\mathbb{T} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$\mathbb{T}^{\dagger} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbb{T}^\dagger\mathbb{T} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

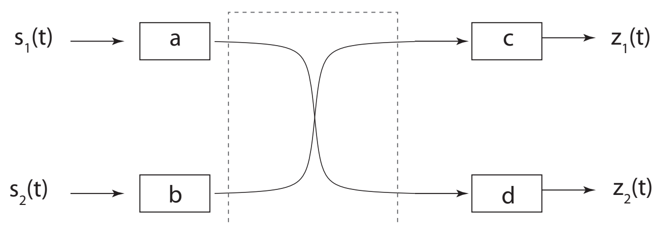
The transformation does not conserve power because it is not unitary.

7.4 *The output of a balanced photodetector*

The output of the balanced photodetector given in (7.3.2) is based on the coupling matrix $\mathbb{T}$ given in (7.1.5). Rederive the output of the balanced photodetector using the alternative form of the coupling matrix $\mathbb{T}$ given in (7.1.2). Comment on the result.

Solution

The quarter-square multiplier given in (7.3.1) requires a coupling matrix of the form given in (7.1.4). This form of coupling matrix can be derived from the symmetric directional coupler matrix given in (7.1.3) by appropriate phase delays on the inputs and outputs as shown in the figure below.



Symmetric directional coupler

Using the symmetric coupling matrix given in (7.1.2) and repeated here

$$\mathbb{T} \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix},$$

and the phase terms a, b, d, d shown in the figure, the coupling matrix

$$\mathbb{T} \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

given in (7.1.4) can be obtained from (7.1.2) when

$$\begin{aligned} a\,c\,s_1 + \mathrm{i}b\,c\,s_2 &= s_1 + s_2 \\ \mathrm{i}a\,d\,s_1 + d\,b\,s_2 &= s_1 - s_2. \end{aligned}$$

This system of equations has a solution when

$$\begin{aligned} b &= -\mathrm{i}a \\ c &= 1/a \\ d &= -\mathrm{i}/a. \end{aligned}$$

Setting $a = 1$ gives $b = -\mathrm{i}$, $c = 1$, and $d = -\mathrm{i}$, which are simply phase shifts on the input and output ports. The overall transform can be expressed as a unitary transformation of the symmetric coupling matrix given in (7.1.3).

7.9 *Lightwave amplifier noise terms*

For a wavelength of $\lambda = 1500$ nm, let $n_{\text{sp}} = 1.25$, $B = 2$ nm, $B_N = 25$ GHz, $G = 30$ dB, $F_N = 5$ dB, and $R = 50\,\Omega$. The output lightwave signal is measured with a photodetector that has a responsivity of 0.8 A/W.

(a) Suppose the stimulated-emission cross section σ_e is twice the absorption cross section σ_a. What is the ratio of the mean upper state density $\overline{N}_2$ to the mean lower state density $\overline{N}_1$ that will produce $n_{\text{sp}} = 1.25$?

Solution
The spontaneous emission noise factor n_{sp} is

$$n_{\text{sp}} = 1.25 = \frac{\sigma_e N_2}{\sigma_e N_2 - \sigma_a N_1}$$

When $\sigma_e = 2\sigma_a$, a ratio of $N_2/N_1 = 2.5$ will produce the desired spontaneous emission factor.

(b) Determine the incident lightwave power P_{in} for which:

(i) The power density spectrum of the shot noise generated by the signal is equal to the power density spectrum generated by thermal noise.

Solution
The thermal power is $k_B T_0 B_N$ where B_N is the noise bandwidth. The signal shot noise power is determined using $2e\mathcal{R}P_{\text{in}}B_N$ (cf. (6.7.8)). This term has units of A^2 where we assume that P_{in} is the optical power **after** the amplifier. Multiplying this term by the load resistance R to convert into Watts gives

$$k_B T_0 = 2eR\mathcal{R}P_{\text{in}}.$$

Solving for P_{in} we have

$$P_{\text{in}} = \frac{k_B T_0}{2R\mathcal{R}e} = \frac{4 \times 10^{-21}}{2 \times 50 \times 0.8 \times 1.6 \times 10^{-19}} = 312.5\,\mu W.$$

(ii) The signal-spontaneous emission noise is equal to the spontaneous-spontaneous emission noise.

Solution
Equating the two terms and using (8.2.36) gives

$$GP_{\text{in}}N_{\text{sp}}B_N + N_{\text{sp}}^2 BB_N$$

where B is the bandwidth of the optical noise-suppressing filter. Solving for P_{in}

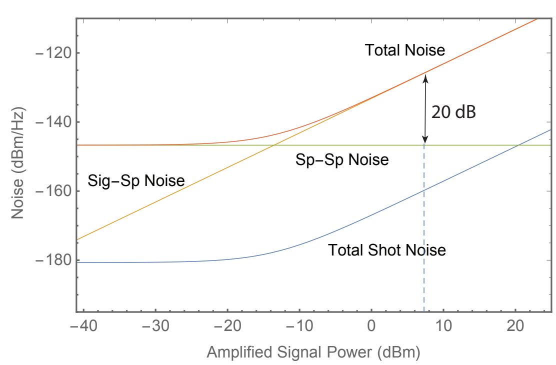
$$P_{\text{in}} = \frac{N_{\text{sp}}B}{G} = \frac{hf\,n_{\text{sp}}(G-1)B}{G} \approx hf\,n_{\text{sp}}\,B \approx \frac{1}{2}hfF_{\text{NP}}B,$$

where the noise figure F_{NP} is defined in (7.7.17). The optical filter bandwidth B in frequency units is $B = \Delta\lambda c/\lambda^2 = 2.67 \times 10^{11}$ GHz. Putting in the other numbers using $f = c/\lambda$ gives

$$P_{\text{in}} = \frac{6.64 \times 10^{-34} \times 3 \times 10^8 \times \sqrt{10} \times 2.67 \times 10^{11}}{2 \times 1.5 \times 10^{-6}} = 55.8\text{ nW} = -42.5\text{ dBm}.$$

(c) For what value of the input power P_{in} does neglecting all the noise terms except the signal-spontaneous emission noise term result in a relative error in the total electrical power that is less than 1%?

Solution
The plot of all the noise terms is shown below. The amplified power that produces < 1% error (or 20 dB) is approximately 7 dBm.



7.11 *Light-emitting diode noise statistics*

A light-emitting diode has a -3 dB spectral bandwidth of 40 nm at 850 nm and mean power of P. This lightwave source is incident on a photodetector with a responsivity of $\mathcal{R} = 0.5$ A/W.

(a) Derive the probability mass function $p(\mathsf{m})$ for the number of photoelectrons over an integration time of T.

Solution

The coherence time is

$$\tau_c = \frac{\lambda^2}{\Delta\lambda c} = 6 \times 10^{-14} \text{ s.}$$

The number of coherence intervals is

$$TB = \frac{T}{\tau_c} = \frac{10^{-9}}{6 \times 10^{-14}} \approx 16,600.$$

The expected generation rate R is

$$\mathsf{R} = \left(\frac{\mathcal{R}}{e}\right) P = \frac{0.5}{1.6 \times 10^{-19}} = 3.125 \times 10^{18} P \qquad \text{counts per second.}$$

Therefore, the mean number E of counts per coherence interval is

$$\mathsf{E} \;=\; \mathsf{R}\tau_c = 6 \times 10^{-14} \times 3.125 \times 10^{18} P \;\approx\; 1.88 \times 10^5 P.$$

In general, the probability distribution for $\underline{\mathsf{m}}$ is a negative binomial distribution (cf. (6.5.11)) characterized by the the expected generation rate R per coherence interval and the number $K = \lceil TB \rceil$ of coherence intervals. For power levels such that the mean E is much less than one and TB is much greater than one, the negative binomial distribution probability distribution reduces to a Poisson distribution.

(b) For what values of $PT = E$ can this source be modeled using a Poisson probability distribution such that the number of photoelectrons is within 5% of the number of photoelectrons calculated using the exact probability distribution?

Solution

Scaling the energy E by the energy per photon to produce the mean number of counts $\mathsf{E} = E/hf$, the negative binomial distribution with mean E and $K = \lceil TB \rceil$ degrees of freedom is

$$p(\mathsf{m}) \quad = \quad \binom{K - 1 + \mathsf{m}}{\mathsf{m}} \left(\frac{1}{1+\mathsf{E}}\right)^K \left(\frac{\mathsf{E}}{1+\mathsf{E}}\right)^{\mathsf{m}} \qquad \text{for } \mathsf{m} = 0, 1, 2, \ldots$$

When $\mathsf{E} \ll 1$ and $K \gg 1$, the negative binomial distribution can be approximated by a Poisson distribution with mean $K\mathsf{E}$.
Using $K \approx TB \approx 16,600$, and determining the relative error for only the value $\mathsf{m} = 0$, a relative error of less than 5% between the negative binomial distribution and the Poisson distribution requires E to be less than 2.488×10^{-3}.

(c) Based on the results of part (b) and in a regime for which the data rate is greater than 1 Mb/s and the power is less than 1 W, is the approximation of $p(\mathsf{m})$ by a Poisson distribution appropriate?

Solution

A data rate of 1 Mb/s corresponding to a signaling interval T equal to 1 μs. For this time interval and the same bandwidth B, the number of coherence intervals is $TB = 1.66 \times 10^7$ with the expected rate for 1 W given by $\mathsf{R} = 3.125 \times 10^{18}$ (see part (a)). Therefore, the mean number of counts E per coherence interval is, from part (a) $\mathsf{E} = 1.88 \times 10^5$. Because this value is much greater than one, approximating the negative binomial distribution by a Poisson distribution may not appropriate for this case.

7.12 *Characteristics of a laser diode*

An idealized laser diode is described by conditions that relate both the lightwave power P_L to the injected current i_{in} and the injected current to the applied voltage V_{in} as follows:

$$P_L = 0.1 i_{\text{in}} \quad \text{for} \quad i_{\text{in}} < 5 \text{ mA} \tag{21a}$$

$$P_L = 1.5 i_{\text{in}} - 7 \quad \text{for} \quad i_{\text{in}} > 5 \text{ mA} \tag{21b}$$

$$i_{\text{in}} = 0.1 e^{V/0.5-1} \quad \text{for} \quad V_{\text{in}} > 0 \text{ volts} \tag{21c}$$

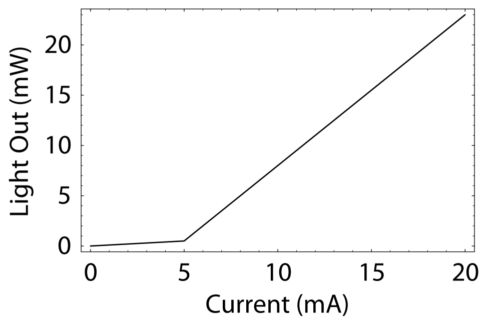
$$i_{\text{in}} = 0 \quad \text{for} \quad V_{\text{in}} < 0 \text{ volts} \tag{21d}$$

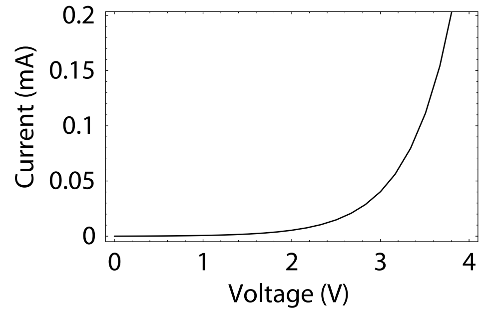
where P_L is the laser power in milliwatts (mW), i_{in} is the current in milliamps (mA), and V_{in} is the voltage in volts (V).

(a) Determine the lasing threshold current and voltage.

Solution

The threshold current is 5 mA. Solving for the voltage when $i_{\text{in}} = 5$ mA, $V = (\log_e(50) + 1)/2$ =2.46 V. Plots of the P_L versus i_{in} and i_{in} versus V are shown in the figure on the next page.





(b) Determine the differential resistance $dV_{\text{in}}/di_{\text{in}}$ at the lasing threshold current and at twice the lasing threshold current.

Solution

Rewriting (21c), the voltage is $V = \frac{1}{2}(\log_e(10i_{\text{in}}) + 1)$. The differential resistance is $(dV/di_{\text{in}}) = \frac{1}{2i_{\text{in}}}1000$, where the factor of 1000 converts mA to A so that the units of resistance are in Ω and not kΩ. The differential resistance at threshold is 100 Ω, and at twice the threshold current, it is 50 Ω.

(c) Determine the ratio of the lightwave power out of the laser to input electrical power in $\left(P_L/(i_{\text{in}}V_{\text{in}})\right)$ for $i_{\text{in}} = 3$ mA and $i_{\text{in}} = 10$ mA.

Solution

For the first case, $i_{\text{in}} = 3$ mA, and the laser is operating below the lasing threshold. Using (21a) for P_L and solving for V in terms of i_{in} in (21c) gives

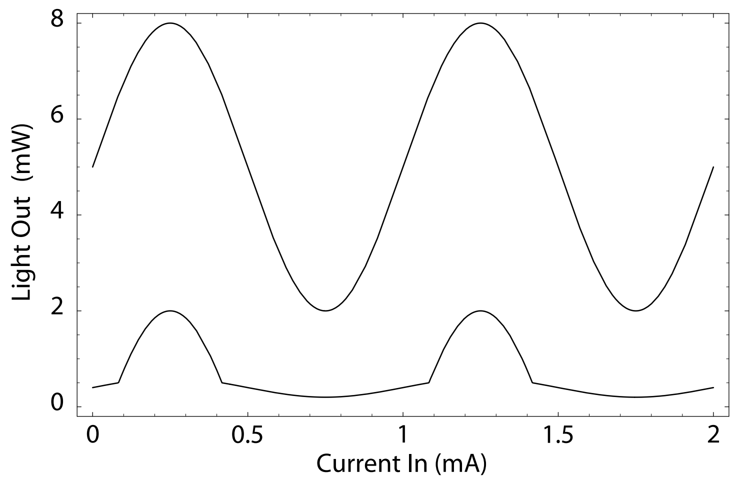
$$\begin{aligned} \frac{P_L}{i_{\text{in}}V_{\text{in}}} &= \frac{0.1i_{\text{in}}}{i_{\text{in}} \times 0.5(\log_e(10i_{\text{in}}) + 1)} \\ &= \frac{0.1(3)}{3 \times 0.5(\log_e(10 \times 3) + 1)} \\ &= 4.54\%. \end{aligned}$$

When $i_{\text{in}} = 10$ mA, the laser is operating above the lasing threshold. Using (21b) gives

$$\begin{aligned} \frac{P_L}{i_{\text{in}}V_{\text{in}}} &= \frac{1.5i_{\text{in}} - 7}{i_{\text{in}} \times 0.5(\log_e(10i_{\text{in}}) + 1)} \\ &= 28.5\%. \end{aligned}$$

(d) A 4 mA peak-to-peak sinusoidal signal plus a bias current i_{bias} is applied to the laser diode. Sketch P_L versus i_{in} for $i_{\text{bias}} = 4$ mA and $i_{\text{bias}} = 8$ mA. Comment on the result.

Solution
The plots of the two modulated lightwave waveforms are shown in the figure below. At a bias of 4 mA, the lower part of the modulating current waveform drives the laser below the lasing threshold and the output lightwave waveform is "clipped", producing distortion. The extinction ratio for this case is equal to 1/10. At 8 mA, the lower peak of the modulating current waveform is above the lasing threshold and the output lightwave waveform is not clipped. The extinction ratio for this case is equal to 1/4.



7.14 *Characterization of a laser diode*

A conventional resonator structure for a laser diode is a Fabry-Perot resonator. This is a resonator constructed using two parallel reflective surfaces. The spacing between the allowed frequencies Δf of a resonator of length d is given by $\Delta f = c_0/2nd$ where c_0 is the speed of light in free space and n is the index of refraction. This value of Δf is called the *free spectral range* of the resonator.

A semiconductor laser is fabricated with a Fabry-Perot resonator of a length $d = 250$ μm and an index $n = 3.5$.

(a) What is the free spectral range of the resonator?

Solution
The free spectral range is

$$\Delta f = \frac{c}{2dn} = \frac{3 \times 10^8}{2(250 \times 10^{-6})(3.5)} = 171.4\,\text{GHz}.$$

(b) Determine the number of possible lasing modes over a -3 dB bandwidth of 0.1 nm.

Solution

The number of possible modes within the specified bandwidth can be determined by expressing the free spectral range in wavelength units. Let $\Delta\lambda_{\text{mode}}$ be the resonator spacing in wavelength units and suppose that the operating wavelength λ is 850 nm. Then

$$\Delta\lambda_{\text{mode}} = \frac{c}{f^2}\Delta f = \frac{\lambda^2}{c}\Delta f = \frac{(850 \times 10^{-9})^2}{3 \times 10^8} \times 171.4 \times 10^9 = 0.4\,\text{nm}.$$

Because the spacing of the resonator modes is four times larger than the 3 dB bandwidth B, there will be at most one lasing mode.

(c) What is the length d of the resonator for which only one mode can lase over this bandwidth?

Solution

For there to be only one lasing mode, $\Delta\lambda_{\text{mode}}$ must be greater than B. Therefore

$$\Delta\lambda_{\text{mode}} = \frac{\lambda^2}{2dn} > B \Rightarrow d < \frac{\lambda^2}{2Bn} = \frac{(850 \times 10^{-9})^2}{2(0.1 \times 10^{-9})(3.5)} \Rightarrow d < 1\,\text{mm}.$$

(d) When the power density spectrum of the relative intensity noise has a constant value of -145 dB/Hz over the frequency range of 0 to 2 GHz, determine the electrical noise power from the relative intensity noise over an integration time T =1 ns for a mean lightwave signal power of 1 mW.

Solution

The noise power from the RIN is given by integrating the noise power density spectrum $N_{\text{RIN}}(f)$ given in (7.8.10). Assuming a responsivity of 1 A/W, this gives

$$\begin{aligned}
\sigma_P^2 &= \int_0^{2\times10^9} N_{\text{RIN}}(f)\mathrm{d}f \\
&= \langle P\rangle^2 \int_0^{2\times10^9} \text{RIN}(f)\mathrm{d}f. \\
&= (10^{-3})^2 \times 2 \times 10^9 \times 10^{-14.5} \\
&= 6.32 \times 10^{-12}\,\text{A}^2
\end{aligned}$$

(e) Compare this noise power to the thermal-noise power generated over the same frequency range. Comment on the result.

Solution
At a temperature of 290K, the available thermal noise power σ^2_{therm} in A^2 over the same bandwidth into a 50 Ω load is

$$\sigma^2_{\text{therm}} = kT_0B/50 = 1.38 \times 10^{-23} \times 290 \times 2 \times 10^9/50 = 1.6 \times 10^{-13}\,\text{A}^2.$$

For this case, the noise from the RIN is larger than the thermal noise.

7.16 *Dark current*

Let μ_{dark} be the stationary dark-current arrival rate within a photodetector.

(a) Using this value, modify the power density spectrum of the emission N_{opt} generated by direct photodetection given in (6.5.3) and repeated here:

$$N_{\text{opt}} \doteq \mathcal{R}P_n\tau_c = \mathcal{R}N_{\text{sp}},$$

to include the effect of the dark current in the photodetector.

Solution
The expression becomes

$$N_{\text{opt}_{\text{total}}} = (\mathcal{R}P_n + e\mu_{\text{dark}})\,\tau_c$$

where the second term is the noise current $i_{\text{noise}} = e\mu_{\text{dark}}$ from the dark-current arrival rate.

(b) Modify the characteristic function $C_r(\omega)$ of the sample value $\underline{r}$ given in (6.7.16) and repeated here:

$$C_r(\omega) = \exp\left(\int_{-\infty}^{\infty} \mathsf{R}(\tau)\left(e^{\mathrm{i}\omega\underline{\mathsf{G}}h(T-\tau)} - 1\right)\mathrm{d}\tau\right)$$

to include the effect of dark-current arrival rate μ_{dark}.

Solution
Viewing $\mathsf{R}(\tau)$ as the expected photogeneration rate, adding the dark current rate gives

$$C_{\underline{r}}(\omega) = \exp\left(\int_{-\infty}^{\infty} \big(\mathsf{R}(\tau) + \mu_{\text{dark}}\big)\big(\exp\left[\mathrm{i}\omega\underline{\mathsf{G}}h(t-\tau)\right] - 1\big)d\tau\right).$$

(c) Determine the mean and the variance of the probability density function for the sample value $\underline{r}$ when the signal photogeneration rate is given by $\mathsf{R}_s(t)$.

Solution

The mean and variance are given by Campbell's theorem given in (6.7.18). Including the dark current arrival rate, these equations are modified to read

$$\langle \underline{r} \rangle = (\mathsf{R}_s(t) + \mu_{\text{dark}}) \circledast h(t)\Big|_{t=T}$$

and

$$\sigma_r^2 = (\mathsf{R}_s(t) + \mu_{\text{dark}}) \circledast h^2(t)\Big|_{t=T}.$$

7.17 *Noise terms*

A lightwave signal generated from a direct-current-modulated laser diode has a power $P = -23$ dBm and a relative intensity noise of -120 dB/Hz. This signal is incident on a photodetector with a responsivity of 0.5 A/W. The output of the photodetector is connected to an electrical amplifier with a noise-equivalent bandwidth $B_N = 15$ GHz and a root-mean-squared thermal-noise current of $\sigma_i = 250$ nA at the input to the electrical amplifier. The amplified signal is then integrated over a time interval T and sampled.

(a) Determine the variance in the sample value due to shot noise.

Solution

The shot noise variance is given by (6.7.5)

$$\begin{aligned}\sigma_r^2 &= 2e\,\langle \underline{i} \rangle\, B_N \\ &= 2e\mathcal{R}PB_N \\ &= 2(1.6 \times 10^{-19}) \times 0.5 \times 10^{-5.3} \times 15 \times 10^9 \\ &= 1.2 \times 10^{-14}\ \text{A}^2.\end{aligned}$$

(b) Determine the variance in the sample value due to relative intensity noise.

Solution

Assuming that the RIN is constant over the noise bandwidth, the variance from the RIN is given by (7.8.10)

$$\begin{aligned}\sigma_{\text{RIN}}^2 &= \mathcal{R}^2P^2\text{RIN}B_N \\ &= \left(0.5 \times 10^{(-5.3}\right)^2 10^{-12} \times 15 \times 10^9 \\ &= 9.42 \times 10^{-14}\ \text{A}^2.\end{aligned}$$

(c) Determine the variance in the sample value due to the thermal noise.

Solution
Squaring the root-mean square current noise $\sigma_i = 250$ nA gives $\sigma_i^2 = 6.25 \times 10^{-14} \quad \text{A}^2$.

(d) Determine the total variance in the sample value.

Solution
For independent noise sources, the variances add. The total variance is 16.9×10^{-14} A^2 or a root-mean squared noise current of 411 nA referred to the input of the electrical amplifier for the specific lightwave power used in this problem.

(e) Determine which noise source has the largest contribution to the overall variance and calculate the relative error in evaluating the root-mean-squared noise when only the most significant noise source is used. Is this a good approximation?

Solution
The relative intensity noise has the largest contribution. It has root-mean squared value equal to 307 nA. The relative error when only this noise noise is used to calculate the root-mean squared current noise is $100 \times (411 - 307)/411$ = 25.22%. This is not a good approximation.

7.21 *Mean and variance of avalanche photodiode probability distribution*

Starting with the characteristic function $C_{\mathsf{m}}(\omega)$ for the output distribution of an avalanche photodiode given by (7.6.3) and repeated here,

$$C_{\mathsf{m}}(\omega) \quad = \quad \exp\left[\frac{\mathsf{w_s}F}{(F-1)^2}\left(1 - \sqrt{1 - 2\mathrm{i}\omega\mathsf{G}(F-1)}\right) - \mathrm{i}\omega\frac{\mathsf{w_s}\mathsf{G}}{F-1}\right],$$

and using (2.2.17), show that the mean of the probability mass function is equal to $\mathsf{E_s}$, and that the variance is given by

$$\sigma_{\mathsf{m}}^2 \quad = \quad \mathsf{G}^2\mathsf{w_s}\left(F + \mathsf{w_s}\right) - \left(\mathsf{w_s}\mathsf{G}\right)^2 \quad = \quad \mathsf{w_s}\mathsf{G}^2F \quad = \quad \mathsf{E_s}\mathsf{G}F,$$

where $\mathsf{G} \doteq \langle \underline{\mathsf{G}} \rangle$ is the mean gain of the avalanche photodiode.

Solution
The expression for the characteristic function is

$$C_{\mathsf{m}}(\omega) \quad = \quad \exp\left[\frac{\mathsf{w_s}F}{(F-1)^2}\left(1 - \sqrt{1 - 2\mathrm{i}\omega\mathsf{G}(F-1)}\right) - \mathrm{i}\omega\frac{\mathsf{w_s}\mathsf{G}}{F-1}\right],$$

Using

$$\langle \underline{\mathsf{m}}^n \rangle = \left| \frac{1}{\mathrm{i}^n}\frac{\mathrm{d}^n}{\mathrm{d}\omega^n} C_{\mathsf{m}}(\omega) \right|_{\omega=0}$$

the mean value is $\langle \underline{\mathsf{m}} \rangle = \mathsf{w_s G}$. The mean square value is $\langle \underline{\mathsf{m}}^2 \rangle = \mathsf{G}^2 \mathsf{w_s} \left(F + \mathsf{w_s} \right)$. Therefore, the variance is

$$\begin{aligned}
\sigma_{\mathsf{m}}^2 &= \langle \underline{\mathsf{m}}^2 \rangle - \langle \underline{\mathsf{m}} \rangle^2 \\
&= \mathsf{G}^2 \mathsf{w_s} \left(F + \mathsf{w_s} \right) - \left(\mathsf{w_s G} \right)^2 \\
&= \mathsf{w_s G}^2 F \\
&= \mathsf{E_s G} F
\end{aligned}$$

where $\mathsf{E_s} = \mathsf{w_s} G$ is the mean number of counts after the internal gain process.

Chapter 8 Selected Solutions

8.2 *Propagation of a chirped gaussian pulse*

An input lightwave pulse $s(t)$ is given as

$$s(t) \quad = \quad Ae^{-t^2/2\sigma_c^2} \tag{22}$$

where $1/\sigma_c^2 = (1 - \mathrm{i}K)/\sigma_{\text{in}}^2$ is complex with the constant K called the chirp parameter. (Note that there is a sign change in this definition of the chirp parameter compared to the original problem statement so as to be consistent with the literature.) The corresponding real-passband lightwave pulse is

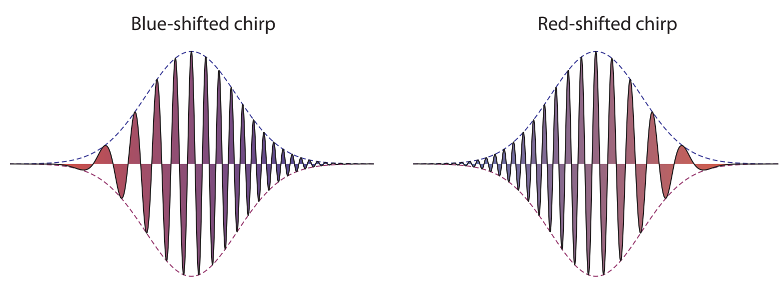
$$\widetilde{s}(t) \quad = \quad Ae^{-t^2/2\sigma_{\text{in}}^2} \cos\Big(2\pi f_c t + \big(K/2\sigma_{\text{in}}^2\big)t^2\Big) \tag{23}$$

with the instantaneous frequency given by

$$f \quad = \quad \frac{1}{2\pi}\frac{d\theta(t)}{dt} \quad = \quad f_c + \left(\frac{K}{4\pi\sigma_{\text{in}}^2}\right)t,$$

where $\theta(t)$ is the argument of the cosine function. When K is positive, increasing time corresponds to increasing frequency. This is called *blue-shifting*. When K is negative, increasing time corresponds to decreasing frequency. This is called *red-shifting*. These two kinds of chirped pulses are shown in the figure below (The ratio of carrier frequency to the spectral width is small enough to show the effect of the chirp.) The pulse passes through a fiber with a transfer function at a distance $z = L$ given by (8.1.3), which is repeated here

$$H(f) \quad = \quad H_0 e^{-\mathrm{i}2\pi\tau f} e^{-\mathrm{i}2\pi^2\beta_2 L f^2}, \tag{24}$$



(a) Determine the input spectral content $S(f)$ of the chirped pulse at $z = 0$.

Solution

The spectral content of the pulse envelope $S(f, z)$ at $z = 0$ is determined by taking the

Fourier transform of complex-baseband pulse given in (22) where $1/\sigma_c^2 = (1 - \mathrm{i}K)/\sigma_{\text{in}}^2$ is complex. Using the Fourier transform pair given by (2.1.37), the frequency spectrum of the chirped pulse is

$$S(f,0) \quad = \quad A\sqrt{2\pi\sigma_c^2}\exp\left(-\frac{(2\pi\sigma_{\text{in}}f)^2}{2(1-\mathrm{i}K)}\right).$$

(b) Determine the root-mean-squared width $\Delta\omega_{\text{rms}}$ of the magnitude of the spectrum $S(f)$ in terms of K and σ_{in}^2.

Solution
The spectrum at $z = 0$ is given by

$$S(f) \quad = \quad A\sqrt{2\pi\sigma_c^2}\exp\left(-\frac{\sigma_c^2(2\pi f)^2}{2}\right) \tag{25}$$

$$= \quad A\left(\frac{2\pi\sigma_{\text{in}}^2}{1-\mathrm{i}K}\right)^{1/2}\exp\left(-\frac{\sigma_{\text{in}}^2(2\pi f)^2}{2(1-\mathrm{i}K)}\right). \tag{26}$$

Separate into real and imaginary parts

$$\exp\left(-\frac{\sigma_{\text{in}}^2(2\pi f)^2}{2(1-\mathrm{i}K)}\right) \quad = \quad \exp\left(-\frac{\sigma_{\text{in}}^2(2\pi f)^2(1+\mathrm{i}K)}{2(1-\mathrm{i}K)(1+\mathrm{i}K)}\right)$$

$$= \quad \exp\left(-\frac{\sigma_{\text{in}}^2(2\pi f)^2}{2(1+K^2)}\right)\exp\left(-\mathrm{i}\frac{\sigma_{\text{in}}^2(2\pi f)^2K)}{2(1+K^2)}\right). \tag{27}$$

The root-mean squared width of magnitude of the spectrum $|S(f,0)|$ can be determined from inspection by writing the real part of (27) in standard form

$$S(f) \quad = \quad A\sqrt{2\pi\sigma_{\text{rms}}^2}e^{-(2\pi f)^2/\sigma_{\text{rms}}^2},$$

so that

$$\sigma_{\text{rms}}^2 = (2\pi\sigma_{\text{in}})^{-1}\sqrt{(1+K^2)}.$$

This is the root-mean squared bandwidth in frequency. The root-mean squared width in angular frequency is multiplied by 2π.

(c) Determine the output spectral content $S_{\text{out}}(f)$ of the chirped pulse at $z = L$.

Solution
Suppose that coherent carrier is used so that the spectrum of the pulse envelope dominates the overall transmitted linewidth σ_λ. Working in the frequency domain, use $S(z,f) =$

$H(z,f)S(z,0)$. Using (26), the spectral content of the chirped pulse after a distance z is then

$$\begin{aligned} S(z,f) &= S(f)H(z,f) \\ &= A\left(\frac{2\pi\sigma_{\text{in}}^2}{1-\mathrm{i}K}\right)^{1/2} \times \\ &\quad \exp\left(-\mathrm{i}2\pi f\tau\right)\exp\left[-\frac{(2\pi f)^2}{2}\left(\frac{\sigma_{\text{in}}^2}{(1-\mathrm{i}K)}+\mathrm{i}\beta_2 z\right)\right] \end{aligned}$$

where the constant H_0 is incorporated into A.
Define the " variance" σ_t^2 after propagation distance of z as

$$\begin{aligned} \sigma_t^2 &\doteq \frac{\sigma_{\text{in}}^2}{(1-\mathrm{i}K)}+\mathrm{i}\beta_2 z \\ &= \frac{\left(\sigma_{\text{in}}^2+(1-\mathrm{i}K)\,\mathrm{i}\beta_2 z\right)}{1-\mathrm{i}K} \\ &= \frac{\left(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z\right)}{1-\mathrm{i}K} \end{aligned}$$

Multiply and divide by σ_t and rearrange

$$\begin{aligned} S(z,f) &= A\sigma_{\text{in}}\sqrt{2\pi}\sigma_t\left(\frac{1}{1-\mathrm{i}K}\right)^{1/2}\left(\frac{1-\mathrm{i}K}{(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z)}\right)^{1/2} \times \\ &\quad \exp\left(-\mathrm{i}2\pi f\tau\right)\exp\left[-\frac{(2\pi f)^2\sigma_t^2}{2}\right] \\ &= \underbrace{\frac{A\sigma_{\text{in}}}{(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z)^{1/2}}}_{\text{constant in frequency}}\underbrace{\left[\sqrt{2\pi}\sigma_t \exp\left(-\mathrm{i}2\pi f\tau\right)\exp\left[-\frac{(2\pi f)^2\sigma_t^2}{2}\right]\right]}_{\text{form for inverse transform}} \end{aligned}$$

The function is now in a form that can be inverse Fourier transformed noting that phase term transforms to time shift

$$\begin{aligned} s(z,t) &= \frac{A\sigma_{\text{in}}}{(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z)^{1/2}}\exp\left(-\frac{(t-\tau)^2}{2\sigma_t^2}\right) \\ &= C(z)\exp\left[-\frac{(1-\mathrm{i}K)\,(t-\tau)^2}{2\,(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z)}\right], \end{aligned}$$

where $C(z)$ is a z dependent constant. Separate $s(z,t)$ to determine the magnitude of the pulse

$$C(z)\exp\left[-\frac{(1-\mathrm{i}K)\,(t-\tau)^2}{2\,(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z)}\right] =$$

$$C(z)\exp\left[-\frac{(1-\mathrm{i}K)\,(t-\tau)^2\left(\sigma_{\text{in}}^2+K\beta_2 z-\mathrm{i}\beta_2 z\right)}{2\left(\sigma_{\text{in}}^2+K\beta_2 z+\mathrm{i}\beta_2 z\right)\left(\sigma_{\text{in}}^2+K\beta_2 z-\mathrm{i}\beta_2 z\right)}\right]$$

Rewriting gives

$$s(z,t) \quad = \quad C(z)\exp\left[-(t-\tau)^2/\Big(2\underbrace{\Big[\Big(\left(\sigma_{\text{in}}^2+K\beta_2 z\right)^2+(\beta_2 z)^2\Big)/\sigma_{\text{in}}^2\Big]}_{\sigma_{\text{out}}^2(z)}\Big)\right] \times \text{phase term.}$$

This expression is in the form of a magnitude and a phase with the magnitude determining the output pulse width.

(d) Show that the square of the ratio of the output timewidth $\sigma_{\text{out}}(z)$ to the input timewidth σ_{in} can be written as

$$\frac{\sigma_{\text{out}}^2(z)}{\sigma_{\text{in}}^2} \quad = \quad \Big(1+K\big(z/L_D\big)\Big)^2+(z/L_D)^2$$

where $L_D = \sigma_{\text{in}}^2/\beta_2$ is the dispersion length (cf. (5.3.23)).

Solution

Using the expression for $\sigma_{\text{out}}^2(z)$ shown above, the mean-squared timewidth of the output pulse can be written as

$$\begin{aligned}
\sigma_{\text{out}}^2(z) &= \frac{\left(\sigma_{\text{in}}^2+K\beta_2 z\right)^2+(\beta_2 z)^2}{\sigma_{\text{in}}^2} \\
&= \sigma_{\text{in}}^2\left[\left(1+K\frac{\beta_2 z}{\sigma_{\text{in}}^2}\right)^2+\left(\frac{\beta_2 z}{\sigma_{\text{in}}^2}\right)^2\right]
\end{aligned}$$

or

$$\frac{\sigma_{\text{out}}^2}{\sigma_{\text{in}}^2} \quad = \quad \Big(1+K\big(z/L_D\big)\Big)^2+(z/L_D)^2\,,$$

where $L_D = \sigma_{\text{in}}^2/\beta_2$ is the dispersion length (cf. (5.3.23)).

(e) Show that when β_2 and K have the same sign, the pulse timewidth increases monotonically with the distance L.

(f) Show that when β_2 and K are opposite in sign, the pulse comes to a "focus" as the pulse propagates in z with the minimum timewidth occurring at a distance given by

$$z_{\text{min}} \quad = \quad \frac{|K|}{1+K^2}L_D.$$

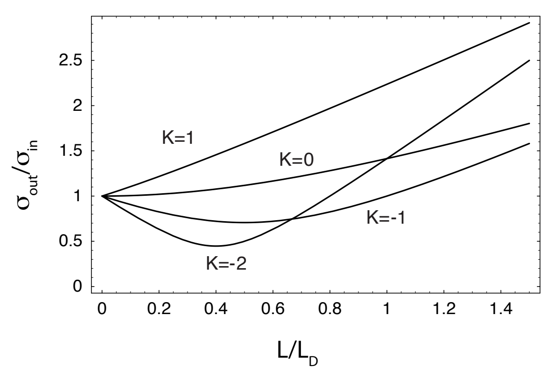
Solution

These two cases differ on the signs of β_2 and K. The choice of signs depends on the time convention for the transfer function $H(f)$ and for the phase chirp. For the sign convention stated in the (modified) problem, the pulse timewidth increases monotonically with the distance L when β_2 and K have same sign. (The original problem statement assumed a different sign convention.) For part (f), for the sign convention chosen for the problem, when β_2 and K have the same sign, the effect of the chirp is to "focus" the pulse in time as the pulse propagates in z with the minimum width occurring at a distance

$$z_{\text{min}} = \frac{|K|}{1+K^2} L_D$$

where L_D is the dispersion distance. At this distance, the imaginary part of the exponential vanishes. For wavelengths longer that the zero dispersion wavelength ($\approx$1.3 μm for standard fiber), the dispersion is anomalous and $\beta_2 < 0$. [1] When $K = 0$ and there is no chirp and $\sigma^2_{\text{out}}(L) = \sigma^2_{\text{in}} \left[1 + (z/z_c)^2\right]$ For large z, the RMS width of the pulse increases linearly with distance.

Using the reciprocal relationship for the RMS width in frequency and time, the RMS width of $a(t, z)$ at z_{min} is given by $\sigma_{\text{out}} = \sigma^2_{\text{in}}/\sqrt{1 + K^2}$. In this case, the RMS width of the pulse at z_{min} is *less* than the input temporal width of σ_{in} leading to "temporal focusing". Plots of the RMS width as function of L/L_D are shown in the figure below. For the sign convention used in the modified problem, when $K \geq 0$, the RMS width increase monotonically. When $K < 0$, the pulse width decreases to a minimum value at z_{min} and then begins to increase after the pulse reaches a minimum pulse width.



[1] Self phase-modulation creates a chirp with $K > 0$ and thus these two effects can cancel for specific pulse shapes and power levels leading to the propagation of stable waveforms called solitons.

8.3 *Variance in the photodetected output*

With the expected power $\langle P\rangle$ collected by direct photodetection held constant, show that the output signal-to-noise ratio (SNR) is proportional to the number M of coherence regions at the output face of the fiber. (Note: the mean and variance of the gamma probability density function are $M\langle P\rangle$ and $M\langle P\rangle^2$, respectively)

Solution

The total power is given by the sum of M independent coherence regions so that

$$\underline{P}_{\text{total}} = \sum_{m=1}^{M} \underline{P}_m,$$

where the set $\{\underline{P}_m\}$ of random variables is identically distributed. Because the random variables are independent, the mean power is $\langle \underline{P}_{\text{total}}\rangle = M\langle \underline{P}_m\rangle$. The electrical SNR is given by

$$\text{SNR} = \frac{M^2\langle \underline{P}_m\rangle}{M\langle \underline{P}_m\rangle} = M.$$

This expresson shows that the SNR increases as the number of independent coherence intervals because of averaging over the coherence regions.

8.4 *Modal noise for a single photodetector*

The output light of a multimode fiber is collected using a single direct photodetector that has an overlap region $\mathcal{A}_{\text{overlap}}$ whose area is equal to the total area of the region $\mathcal{A}_{\text{fiber}}$ of the output face of the fiber including the core and the cladding.

(a) Is there modal noise when there is no mode-selective attenuation? Explain.

Solution

No. When all of the signal power is collected from every mode that contains power, there is no modal noise.

(b) Is there modal noise when the photodetector is misaligned and collects only a portion of the power in the fiber and there are no other mode-selective attenuation mechanisms? Explain.

Solution

Yes. For this case, a random portion of the incident lightwave power is coupled into the photodetector as the speckle pattern randomly shifts across the output aperture. This form of mode-dependent loss is modal noise.

8.6 *Output pulse for a fiber that supports two modes*

Consider a fiber that supports two spatial modes. The output lightwave pulse in the first mode before photodetection is a unit-amplitude gaussian pulse with unit variance. The output lightwave pulse before photodetection in the second mode is a unit-amplitude gaussian pulse also with unit variance, but is delayed in time by a value equal to one-half the variance. Determine an expression for the electrical signal energy E when:

(a) The pulses in each mode are noncoherent.

Solution

When the pulses are noncoherent, the directly photodetected electrical signal $r(t)$ is is the sum of the power in each pulse and is given by (1.2.4)

$$\begin{aligned} r(t) &= \mathcal{R}\left(P_1(t) + P_2(t)\right) \\ &= \frac{\mathcal{R}}{2}\left(e^{-t^2} + e^{-(t-1/2)^2}\right), \end{aligned}$$

where $P_j(t) = |s_j(t)|^2/2$ for $j = 1, 2$, $s_1(t) = e^{-t^2/2}$, $s_3(t) = e^{-(t-1/2)^2/2}$, and $\mathcal{R}$ is the responsivity. The electrical energy E over an interval T is

$$\begin{aligned} E &= \int_T r^2(t)\mathrm{d}t \\ &= \frac{\mathcal{R}^2}{4}\int_T \left(e^{-t^2} + e^{-(t-1/2)^2}\right)^2 \mathrm{d}t \\ &= \frac{\mathcal{R}^2}{4}\int_T \left(e^{-2t^2} + 2e^{-t^2-(t-1/2)^2} + e^{-2(t-1/2)^2}\right) \mathrm{d}t \end{aligned}$$

(b) The pulses in each mode are coherent.

Solution

When the pulses are coherent, the directly photodetected electrical signal $r(t)$ is is the square of the sum of the amplitudes in each pulse so that

$$\begin{aligned} r(t) &= \frac{\mathcal{R}}{2}\left(e^{-t^2/2} + e^{-(t-1/2)^2/2}\right)^2 \\ &= \frac{\mathcal{R}}{2}\left(e^{-t^2} + 2e^{-\frac{t^2}{2}-\frac{1}{2}(t-1/2)^2} + e^{-(t-1/2)^2}\right), \end{aligned}$$

showing additional cross term. The corresponding electrical energy E over an interval T is

$$
\begin{aligned}
E &= \int_T r^2(t)\mathrm{d}t \\
&= \frac{\mathcal{R}^2}{4}\int_T \left(e^{-t^2/2} + e^{-(t-1/2)^2/2}\right)^4 \mathrm{d}t \\
&= \frac{\mathcal{R}^2}{4}\int_T \left(e^{-2t^2} + 4e^{-\frac{3t^2}{2}-\frac{1}{2}(t-1/2)^2} + 6e^{-t^2-(t-1/2)^2} + 4e^{-\frac{t^2}{2}-\frac{3}{2}(t-1/2)^2} + e^{-2(t-1/2)^2}\right)\mathrm{d}t
\end{aligned}
$$

(c) Comment on the result.

Solution

The coherent cross terms can cause either constructive interference between the pulses depending on the relative phase between the pulses.

8.7 *Amplitude-phase coupling in a dispersive fiber*

Suppose that a lightwave signal $s(t)$ at the input to a dispersive fiber is sinusoidally phase-modulated so that

$$s(t) = e^{\mathrm{i}\mu\sin(2\pi f_m t)},$$

where μ is the modulation index, and f_m is the modulation frequency with period $T = 1/f_m$. This periodic signal can be expressed in terms of an exponential Fourier series given by

$$s(t) = e^{\mathrm{i}\mu\sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} F_n e^{\mathrm{i}n2\pi f_m t},$$

with the Fourier series coefficients F_n given by $J_n(\mu)$, the Bessel function of the first kind and order n.

(a) Derive an expression for the output lightwave signal $s(t)$ at a distance L in terms of the Fourier series coefficients and the complex-baseband transfer function given in (8.1.3).

Solution

Given that the input signal is already expressed in terms of a superposition of exponential functions of the form of $e^{\mathrm{i}n2\pi f_0 t}$, the output can immediately be written as

$$
\begin{aligned}
r(t) &= \sum_{n=-\infty}^{\infty} J_n(M)e^{\mathrm{i}n2\pi f_0 t}H(nf_0) \\
&= H_0\sum_{n=-\infty}^{\infty} J_n(M)e^{\mathrm{i}n2\pi f_0(t-\tau)}e^{-\mathrm{i}\pi\beta_2(nf_0)^2L},
\end{aligned}
$$

where $H(f)$ is given in (8.1.3).

(b) By equating terms of the same frequency, determine an expression for the output lightwave power $P \doteq |s(t)|^2/2$ at frequency f_m.

Solution

The form of the photodetected signal is the product of two summation with differing indices. Therefore,

$$\begin{aligned} i(t) \quad = \quad \frac{1}{2}|r(t)|^2 \quad &= \quad H_0 \sum_{n=-\infty}^{\infty} J_n(M) e^{\mathrm{i}n2\pi f_0(t-\tau)} e^{-\mathrm{i}\pi\beta_2(nf_0)^2 L} \\ &\times \sum_{m=-\infty}^{\infty} J_m(M) e^{\mathrm{i}m2\pi f_0(t-\tau)} e^{-\mathrm{i}\pi\beta_2(mf_0)^2 L} \end{aligned}$$

The term at f_0 is generated when the difference in the two indices is equal to one. This gives

$$\begin{aligned} i(t) \quad &\propto \quad J_0(M)J_1(M)\cos\left(\pi\beta_2 f_0^2 L\right) \\ &+ \quad J_1(M)J_2(M)\cos\left(3\pi\beta_2 f_0^2 L\right) + ... \end{aligned}$$

where the factor of three is from $2^2 - 1^2$. The problem shows that a dispersive medium such as a fiber will convert a constant amplitude phase modulated signal into amplitude fluctuations at the output.

Chapter 9 Selected Solutions

9.4 *Exact and approximate thresholds*

(a) Derive the threshold expression given in (9.5.21).

Solution

Rewrite the equation as

$$\sigma_0^2 (r - s_1)^2 - \sigma_1^2 (r - s_0)^2 + 2\sigma_1^2\sigma_0^2 \left(\log_e (p_0\sigma_1) - \log_e (p_1\sigma_0)\right) \quad = \quad 0.$$

Using the quadratic formula and noting the term $\left(s_0\sigma_1^2 - s_1\sigma_0^2\right)^2 - \left(\sigma_0^2 - \sigma_1^2\right)\left(s_1^2\sigma_0^2 - s_0^2\sigma_1^2\right)$ inside the square root function can be factored into $(s_0 - s_1)^2\,\sigma_0^2\sigma_1^2$ gives (9.5.21).

(b) Show that for $(s_1 - s_0)^2$ much larger than $2(\sigma_1^2 - \sigma_0^2)\log_e(\sigma_1/\sigma_0)$, that $p_{1|0}$ and $p_{0|1}$ are approximately equal, which demonstrates that the channel is approximately a binary symmetric channel.

Solution

Using the approximation stated in the problem, the second term inside the square-root function can be neglected. Using the larger of the two thresholds then gives

$$\begin{aligned}
\Theta &= \frac{s_0\sigma_1^2 - s_1\sigma_0^2 + \sigma_1\sigma_0(s_1 - s_0)}{\sigma_1^2 - \sigma_0^2} \\
&= \frac{s_1\sigma_0(\sigma_1 - \sigma_0) + s_0\sigma_1(\sigma_1 - \sigma_0)}{(\sigma_1 + \sigma_0)(\sigma_1 - \sigma_0)} \\
&= \frac{s_1\sigma_0 + s_0\sigma_1}{\sigma_1 + \sigma_0}
\end{aligned}$$

which is the threshold Θ that produces for a binary symmetric channel when the variances are equal and prior $p_{1|0} = p_{0|1} = 1/2$ is equiprobable.

9.6 *Gaussian probability density function with signal-independent and signal-dependent variances*

Let the expected sample value s_1 when a mark is transmitted be equal to 200. Let the expected sample value s_0 when a space is transmitted be equal 20. The system has additive signal-independent gaussian noise characterized by $\sigma^2 = 900$, and signal-dependent noise characterized by $\sigma_\ell^2 = s_\ell$, where s_ℓ is the expected sample value. Using (9.5.27) and (9.5.28) determine the following:

(a) The probability of a detection error p_e and the threshold Θ when only the signal-dependent additive noise term is included.

Solution

When only signal-dependent shot noise is included, the $\mathcal{Q}$ parameter is

$$\mathcal{Q} \quad = \quad \frac{s_1 - s_0}{\sqrt{s_1} + \sqrt{s_0}} \quad = \quad \frac{200 - 20}{\sqrt{200} + \sqrt{20}} \quad = \quad 9.67.$$

Using (9.5.26) we have

$$p_e \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{\mathcal{Q}}{\sqrt{2}}\right) \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{9.67}{\sqrt{2}}\right) \quad = \quad 2 \times 10^{-22}.$$

The threshold is the geometric mean so that

$$\Theta \quad = \quad \sqrt{200 \times 20} \quad = \quad 20\sqrt{10}.$$

(b) The probability of a detection error p_e and the threshold Θ when only the signal-independent additive noise term is included.

Solution

When only signal-dependent shot noise is considered then

$$\mathcal{Q} \quad = \quad \frac{s_1 - s_0}{2\sigma} \quad = \quad \frac{200 - 20}{60} \quad = \quad 3,$$

and

$$p_e \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{\mathcal{Q}}{\sqrt{2}}\right) \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{3}{\sqrt{2}}\right) \quad = \quad 1.35 \times 10^{-3}.$$

The threshold is the arithmetic mean so that

$$\Theta \quad = \quad \frac{200 + 20}{2} \quad = \quad 110.$$

(c) The probability of a detection error p_e and the threshold Θ when both noise terms are included.

Solution

When both noise sources are considered

$$\mathcal{Q} \quad = \quad \frac{s_1 - s_0}{\sqrt{\sigma^2 + s_1} + \sqrt{\sigma^2 + s_0}} \quad = \quad \frac{200 - 20}{60} \quad = \quad 2.83,$$

and

$$p_e \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{\mathcal{Q}}{\sqrt{2}}\right) \quad = \quad \frac{1}{2}\text{erfc}\left(\frac{2.83}{\sqrt{2}}\right) \quad = \quad 2.29 \times 10^{-3}.$$

The threshold is given by (9.5.25)

$$\begin{aligned} \Theta &= \frac{\sigma_1 s_0 + \sigma_0 s_1}{\sigma_1 + \sigma_0} \\ &= \frac{s_0\sqrt{\sigma^2 + s_1} + s_1\sqrt{\sigma^2 + s_0}}{\sqrt{\sigma^2 + s_0} + \sqrt{\sigma^2 + s_0}} \quad = \quad \frac{20\sqrt{900 + 200} + 200\sqrt{900 + 20}}{\sqrt{900 + 200} + \sqrt{900 + 20}} \quad = \quad 106, \end{aligned}$$

which is slightly less than the threshold derived using only additive noise.

(d) Based on this analysis, which noise source is more significant?

Solution
The system is additive-noise limited.

9.8 *Thresholds for a multilevel system*

(a) A multilevel system with L levels, with $\sigma_\ell = \sigma$ being a constant, is indexed by ℓ. Show that, for this system, $\overline{\gamma}_\ell$ is a constant and that the minimum probability of a detection error p_e is achieved for uniformly spaced signal levels s_ℓ.

Solution
Start with (9.6.8), which is repeated here

$$p_e \quad = \quad \frac{L-1}{L}\text{erfc}\left(\sqrt{\overline{\gamma}/2}\right),$$

where $\overline{\gamma} = \mathcal{Q}^2$. Suppose for simplicity that $L = 4$. The arguments R_i of the error functions can be written as

$$p_e \quad = \quad \frac{3}{4}\left(\text{erfc}(R_1) + \text{erfc}(R_2) + \text{erfc}(R_3)\right), \tag{28}$$

with a constraint $R_1+R_2+R_3 = K$ where K is related to the average power. To determine the minimum probability p_e of a detection error in terms of the mean signal levels, use Lagrange multipliers. Taking the gradient of (28) with respect to $\{R_1, R_2, R_3\}$ generates an equation for each of the three components R_ℓ. Each of these equations equals the Lagrange multiplier λ. Including the constraint of $R_1 + R_2 + R_3 = K$ gives four equations and four unknowns (R_1, R_2, R_3, λ)

$$\begin{aligned}
-\frac{2}{\sqrt{\pi}}e^{-R_1^2} &= \lambda \\
-\frac{2}{\sqrt{\pi}}e^{-R_2^2} &= \lambda \\
-\frac{2}{\sqrt{\pi}}e^{-R_3^2} &= \lambda \\
R_1 + R_2 + R_3 &= K,
\end{aligned}$$

where $\mathrm{d}/\mathrm{d}x\,\mathrm{erfc}(x) = -2e^{-x^2}/\sqrt{\pi}$ has been used. The symmetric nature of these equations gives the solution as $R_1 = R_2 = R_3 = K/3$ showing the minimum p_e is achieved for equal spacing between the levels.

(b) Now consider an ideal shot-noise-limited system with $\sigma_\ell = \sqrt{s_\ell}$, supposing that the square root $\sqrt{s_\ell}$ of the expected signal levels are uniformly spaced. Show that for this system $\overline{\gamma}_\ell$ is again a constant that does not depend on ℓ.

Solution

For signal-dependent noise, the expression for the argument error function is proportional to $\mathcal{Q}$ as given in (9.7.2) and repeated here

$$\begin{aligned}
\mathcal{Q}_\ell &= \frac{s_{\ell+1} - s_\ell}{\sigma_{\ell+1} + \sigma_\ell} &&= \frac{s_{\ell+1} - s_\ell}{\sqrt{s_{\ell+1}} + \sqrt{s_\ell}} \\
&= \frac{(\sqrt{s_{\ell+1}} + \sqrt{s_\ell})(\sqrt{s_{\ell+1}} - \sqrt{s_\ell})}{\sqrt{s_{\ell+1}} + \sqrt{s_\ell}} &&= \sqrt{s_{\ell+1}} - \sqrt{s_\ell}.
\end{aligned}$$

Given that the problem states that the square roots are uniformly spaced, the term $\sqrt{s_{\ell+1}} - \sqrt{s_\ell}$ is a constant. Therefore $\mathcal{Q}$ is a constant and $\overline{\gamma}_\ell = \mathcal{Q}^2$ is a constant that does not depend on the specific level ℓ.

(c) Show that the uniform spacing of the square root of the signal levels for a shot-noise-limited system produces a minimum probability of a detection error p_e.

Solution

The same Lagrange multiplier method used in part (a) can be used for this problem.

9.14 *Sensitivity of the probability of a detection error*

This problem quantifies the sensitivity of the probability of a detection error p_e to the value of $\overline{\gamma}$.

(a) Using (9.5.30), determine the value of $\overline{\gamma}$ that produces $p_e = 10^{-9}$.

Solution
Using (9.5.26) we have

$$p_e \quad = \quad \frac{1}{2}\text{erfc}\left(\sqrt{\overline{\gamma}/2}\right).$$

Setting $p_e = 10^{-9}$ and solving, the value of $\overline{\gamma}$ is 36.

(b) Determine p_e when the value of $\overline{\gamma}$ determined in part (a) is halved, and comment on the result.

Solution
For $\overline{\gamma} = 36/2 = 18$, we have

$$p_e \quad = \quad \frac{1}{2}\text{erfc}\left(\sqrt{18/2}\right) \quad = \quad 1.1 \times 10^{-5}.$$

The probability of a detection error changes by over four orders of magnitude.

(c) Let $\overline{\gamma} = \overline{\gamma}_0 + \delta\overline{\gamma}$. Expand the approximate expression for $p_e(\overline{\gamma})$ given in (9.5.30) keeping only terms of order zero and order δ.

Solution
Using (2.2.20), the approximate expression for p_e is

$$p_e \quad \approx \quad \frac{1}{\sqrt{\overline{\gamma}\pi/2}}e^{-\overline{\gamma}/2}.$$

Ignoring the scaling factor in front of the exponential,

$$e^{-(\overline{\gamma}_0+\delta\overline{\gamma})/2} \quad = \quad p_e(\overline{\gamma}_0)\delta p_e,$$

where δp_e is simply

$$\delta p_e \quad = \quad e^{-\delta\overline{\gamma}/2},$$

where it is assumed that δ is negative so that it increases the error probability.

(d) Using this expansion and a nominal value of $\overline{\gamma}_0 = 9$, determine the change in the probability of a detection error when the value of $\overline{\gamma}$ changes by 5%.

Solution

Using $\delta\gamma = 9(0.05) = 0.45$ gives the error as

$$\delta p_e = e^{0.45/2} = 25\%,$$

The exponential sensitivity of the error rate with respect to the argument of the erfc function means that if there are multiple terms to evaluate for the total probability of a detection error, the term with the smallest value will dominate unless the terms are nearly identical or there is a large multiplicity of the same type of term.

9.16 *Unequal prior probabilities*

Consider two systems. The first system determines the threshold knowing the priors by using the ratio of the posterior probability density functions $u(r)$ given in (9.5.6). The second system determines the threshold using the likelihood ratio $\lambda(r)$ based on an equiprobable prior.

(a) Derive an expression for the relative error in the probability of a detection error using $\lambda(r)$ compared to using $u(r)$ as function of the ratio of the prior probabilities p_0/p_1 when the two conditional probability density functions are gaussian probability density functions with unit variance.

Solution

The exact threshold Θ is given by the solution to (9.5.21) and is repeated here setting $\sigma_1 = \sigma_0 = 1$

$$\tfrac{1}{2}\big((r-s_1)^2 - (r-s_0)^2\big) + \log_e(p_0) - \log_e(p_1) = 0.$$

Because $\sigma_1 = \sigma_0 = 1$, there is only single threshold given by

$$\begin{aligned} \Theta &= \frac{s_1+s_0}{2} + \frac{\log_e \mathfrak{r}}{s_1 - s_0} \\ &= \Theta_0 + \delta\Theta \end{aligned}$$

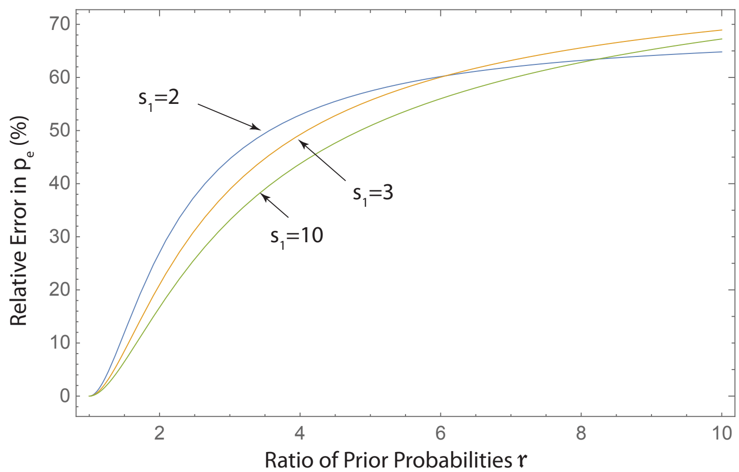
where $\mathfrak{r} = p_0/p_1$ is the ratio of the priors (cf. (9.5.11)), Θ_0 is the threshold for an equiprobable prior and $\delta\Theta$ is the change in the threshold because of the unequal probabilities. This threshold is then used in (9.5.23) to derive the corresponding conditional probabilities $p_{1|0}$ and $p_{0|1}$ of a detection error. The total probability of a detection error is given by (9.5.25). Using $p_1 = 1/(\mathfrak{r}+1)$, $p_0 = \mathfrak{r}/(\mathfrak{r}+1)$, and $\sigma_1 = \sigma_0 = 1$ gives

$$p_e = \frac{1}{2(\mathfrak{r}+1)}\text{erfc}\left(\frac{\Theta - s_0}{\sqrt{2}}\right) + \frac{\mathfrak{r}}{2(\mathfrak{r}+1)}\text{erfc}\left(\frac{s_1 - \Theta}{\sqrt{2}}\right).$$

(b) Plot the relative error over the interval $1 < (p_0/p_1) < 10$ for s_0 equal to one and s_1 equal to: (i) 2, (ii) 3, and (iii) 6. Comment on the result with regard to the dependence of the relative error on the prior probability ratio and the signal-to-noise-ratio.

Solution

The relative error using the approximate threshold Θ_0 based on equiprobable prior as compared to the exact threshold Θ based on the posterior probability is shown in the figure below.



The error is zero for $p_0/p_1 = 1$ and monotonically increases for any value of s_1. The curves for $s_1 = 2$ and $s_1 = 3$ cross over because for these small mean values, an accurate calculation of the probability of a detection error requires the use of both detection regions (cf. Figure 9.11).

9.17 *Local oscillator power required for shot-noise-limited performance*

A phase-synchronous demodulator uses a photodetector with a responsivity $\mathcal{R} = 1$ A/W at $1.5\ \mu$m and a dark current of 1 nA. The photodetector is connected to a preamplifier with a root-mean-squared current noise density spectrum $\sigma = 1$ pA/$\sqrt{\text{Hz}}$ at the input to the amplifier. If $B_N = 0.75R$ where R is the data rate in bits/s and B_N is the noise bandwidth, derive an expression for the required local oscillator power as a function of the data rate R so that the sum of the electrical thermal noise and the dark current noise is one percent of the shot noise generated by the local oscillator.

Solution

The thermal noise current variance σ_i^2 in units of A^2 is given by

$$\sigma_i^2 \quad = \quad \sigma^2 B_N \ \mathrm{A}^2$$

The dark noise current variance σ_d^2 in units of A^2 is

$$\sigma_d^2 \quad = \quad 10^{-18} \ \mathrm{A}^2$$

because the variance is equal to the mean value for a Poisson random variable. The shot noise variance σ_{shot}^2 in units of A^2 is

$$\begin{aligned} \sigma_{\text{shot}}^2 &= 2e\langle i\rangle B_N \\ &= 2e\mathcal{R}P_{LO}B_N, \end{aligned}$$

where P_{LO} is the mean local oscillator power. Solving for the LO power gives

$$P_{LO} \quad = \quad \frac{\sigma_{\text{shot}}^2}{2e\mathcal{R}B_N}.$$

Setting the shot noise variance σ_{shot}^2 equal to 100 times the sum of the thermal noise variance and the dark current variance gives $\sigma_{\text{shot}}^2 = 100(\sigma_i^2 + \sigma_d^2) = 100(\sigma^2 B_N + 10^{-18})$. Substituting the numerical values gives

$$P_{LO} \quad = \quad \frac{100(10^{-24}(0.75R) + 10^{-18})}{2(1.6 \times 10^{-19})(0.75R)}$$

where $B_N = 0.75R$ where R is the data rate in bits/s. For data rates greater than 1 Gb/s, the thermal noise from the amplifier dominates the dark current from the photodetector so that

$$P_{LO} \quad \approx \quad \frac{100(10^{-24})}{2(1.6 \times 10^{-19})} \quad \approx \quad 3.125 \times 10^{-4}\,\mathrm{W} \quad \approx \quad -5\,\mathrm{dBm}.$$

For the conditions stated in this problem, the required local oscillator power is independent of the data rate R.

9.19 *Detection thresholds*

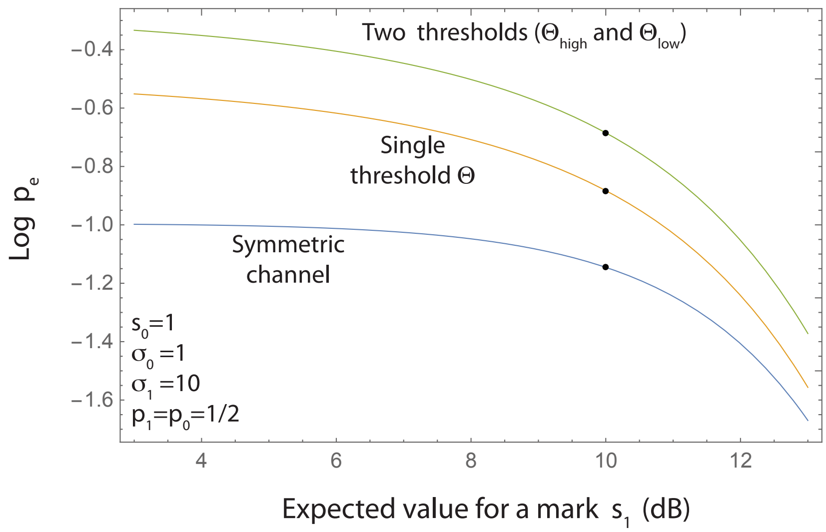
This problem compares the probability of a detection error based on three different methods of detection: the first uses the two thresholds defined by the solutions to (9.5.21), the second uses only the larger of the two value given in (9.5.22), and the third uses a threshold chosen to produce a binary symmetric channel with p_e given by (9.5.30).

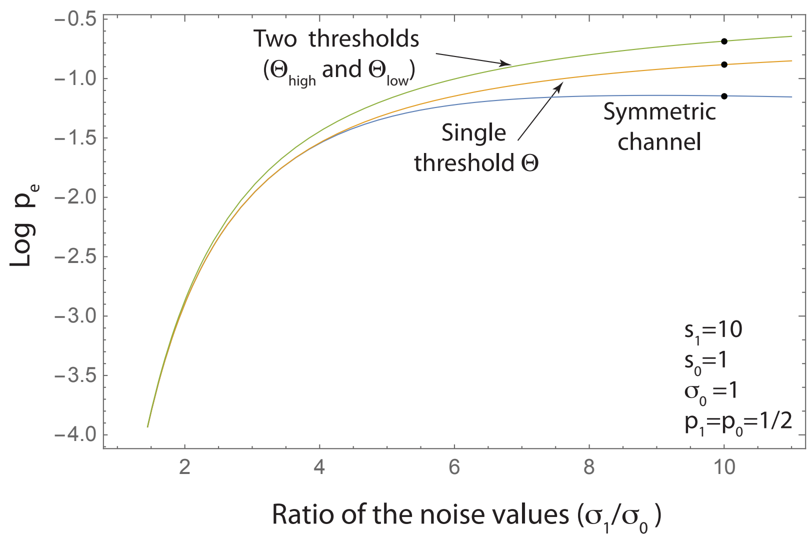
(a) Let $s_0 = 0$, $\sigma_0 = 1$, $\sigma_1 = 10$, and $p_0 = p_1 = 1/2$. Plot the logarithm of p_e versus the

logarithm of the expected value of a mark s_1 for values of s_1 up to 40. Comment on the result.
(b) Let $s_0 = 0$, $s_1 = 10$, $\sigma_0 = 1$, and $p_0 = p_1 = 1/2$. Plot the logarithm of p_e versus the ratio σ_1/σ_0 from values slightly greater than one up to ten. Comment on the result.
(c) Based on these plots, comment on the range of validity of modeling the channel as a binary symmetric channel.

Solution

The difference between the probability of a detection error p_e for detection based on using two optimal thresholds, detection based on using one threshold, and detection based on using a threshold chosen to produce a binary symmetric channel is shown in the figure on the next page as a function of the expected value r_1 for a mark. The lower plot is the same set of curves plotted as a function of the ratio σ_1/σ_0. The marked points are the same for each curve. The difference between the three methods of detection is most pronounced when the variances σ_0^2 and σ_1^2 of the two probability density functions are significantly different and the expected signal levels s_0 and s_1 are small. Referring to the lower plot, all three methods of detection produce the same p_e as the variances become comparable as is evident in the figure. All three methods also produce the same p_e for conditions that produce a binary symmetric channel. These conditions are satisfied by nearly all current lightwave communication systems and thus detection probabilities based on a binary symmetric channel is widely used.


Two thresholds (Θ_{high} and Θ_{low})
Single threshold Θ
Symmetric channel
$s_0=1$
$\sigma_0=1$
$\sigma_1=10$
$p_1=p_0=1/2$
Log p_e
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
4
6
8
10
12
Expected value for a mark s_1 (dB)


Two thresholds (Θ_{high} and Θ_{low})
Single threshold Θ
Symmetric channel
$s_1=10$
$s_0=1$
$\sigma_0=1$
$p_1=p_0=1/2$
Log p_e
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
2
4
6
8
10
Ratio of the noise values (σ_1/σ_0)

Chapter 10 Selected Solutions

10.1 *The photocharge and the electrical energy in a pulse*

For direct photodetection, the photocharge W in an electrical pulse $p(t)$ is given by

$$W \doteq \int_{-\infty}^{\infty} p(t)\mathrm{d}t,$$

and is directly proportional to the lightwave energy with the responsivity $\mathcal{R}$ (cf. Table 6.2) as the proportionality constant. The electrical energy in the same pulse for a unit resistance R is

$$E \doteq \int_{-\infty}^{\infty} p^2(t)\mathrm{d}t.$$

Using $\mathcal{R} = 1$ and $R = 1$, compare the lightwave signal energy and the electrical signal energy for the pulses following:

(a) $p(t) = A\,\mathrm{rect}(t)$

Solution

$$W = \int_{-\infty}^{\infty} A\,\mathrm{rect}(t)\mathrm{d}t = A \qquad E = \int_{-\infty}^{\infty} A\,\mathrm{rect}^2(t)\mathrm{d}t = A^2.$$

(b) $p(t) = A\,\mathrm{sinc}(t)$

Solution

$$W = \int_{-\infty}^{\infty} A\,\mathrm{sinc}(t)\mathrm{d}t = A \qquad E = \int_{-\infty}^{\infty} A\mathrm{sinc}^2(t)\mathrm{d}t = A^2.$$

(c) $p(t) = \frac{A}{\sqrt{2\pi}},e^{-t^2/2}$

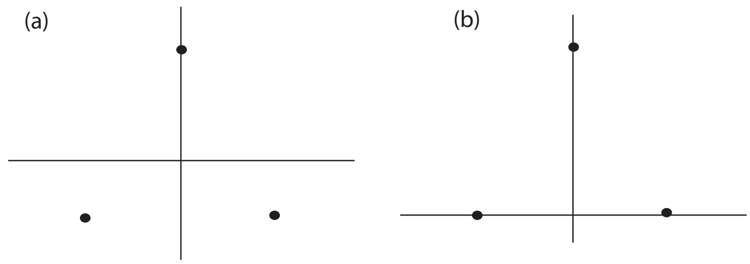
Solution

$$W = \int_{-\infty}^{\infty} \tfrac{A}{\sqrt{2\pi}}e^{-t^2/2}\mathrm{d}t = A \qquad E = \int_{-\infty}^{\infty} \left(\tfrac{A}{\sqrt{2\pi}}e^{-t^2/2}\right)^2\mathrm{d}t = \frac{A^2}{2\sqrt{\pi}}.$$

Examining the three pulses there is no general relationship between the lightwave signal energy and the electrical signal energy.

10.3 *The effect of a constant bias signal on the optimal threshold*

Consider the two three-point signal constellations shown below. In each constellation, the three points form an equilateral triangle with side of length d.



(a) Determine the mean symbol energy E in terms of d when each of the three signal points is equidistant from the origin as part (a) of the figure. Repeat for part (b). In this case, the three signal points do not have the same energy.

Solution

For the constellation shown in part (a), every symbol has the same energy, which is $E = d/\sqrt{3}$. For figure (b), the three signal points do not have the same energy. The two points on the horizontal axis have an energy of $d/2$, and the third point has an energy of $\sqrt{3}d/2$, which is simply the height of the triangle.

(b) Partition the plane for each constellation into three optimal decision regions when the noise is additive white gaussian noise.

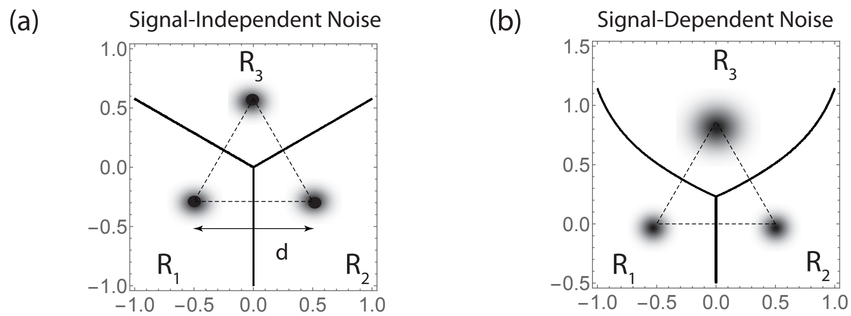
Solution

The partitioning is shown in part (a) of the figure on the next page. For the second case, the decision regions are displaced, but are still straight lines.

(c) Partition the plane for the constellation shown in part (a) of the figure into three optimal decision regions for the case of zero-mean gaussian noise with a variance that is proportional to the mean signal. Compare your answer with the results of part (b) of this problem.

Solution

Because the signal point are equidistant from the origin, the variance of each of these probability density functions is the same. Therefore, the partitions for the decision regions do not change and are shown in part (a) of the figure on the next page.



(d) Repeat part (c) using figure (b) and sketching the approximate decision regions. Compare these regions with the results for a white additive gaussian noise channel. Are the decision regions the same for both signal constellations?

Solution
In this case, the distance of the the point along the vertical axis is $\sqrt{3}$ larger than the distance the other two points along the horizontal axis. This leads to a larger noise variance for this point. Consequently, the decision regions are not straight lines. These decision regions are shown in part (b) of the figure for the solution.

10.8 *Multilevel intensity modulation*

A four-level Gray-coded intensity-modulated system is designed to achieve a probability of detection error p_e. It has a mean background noise term s_0 and a signal-independent noise variance σ^2. Using (9.6.8), determine:

(a) The required value for $\mathcal{Q}$.

Solution
The levels are determined iteratively. Using the symbol error rate stated in (9.6.8) gives

$$p_e \quad = \quad \frac{(L-1)}{L}\mathrm{erfc}(\mathcal{Q}/\sqrt{2}).$$

Setting $L = 4$ and solving for $\mathcal{Q}$ (or $\overline{\gamma}^2$) we have

$$\mathcal{Q} \quad = \quad \sqrt{2}\mathrm{erfc}^{-1}(4p_e/3).$$

(b) The required expected signal levels s_1 through s_3 in terms of s_0, σ^2, and p_e.

Solution
Under the constraint that noise is signal independent, $Q_\ell = (s_{\ell+1} - s_\ell)/2\sigma$ is the same for all ℓ (See Problem 9.8). Starting with s_0 the remaining values s_i are given by

$$s_{\ell+1} = 2\sqrt{2}\sigma \text{erfc}^{-1}(4p_e/3) + s_\ell,$$

for $\ell = 0, 1, 2$.

(c) The threshold values Θ_1 through Θ_3.

Solution
Using (9.6.3a) with equal variances for all ℓ gives

$$\Theta_{\ell+1} = \frac{1}{2}\left(s_{\ell+1} + s_\ell\right),$$

for $\ell = 0, 1, 2$.

(d) The expected number of photoelectrons m per symbol.

Solution
The received sample r_ℓ is the photocharge W in an interval T. Therefore, the average number of photoelectrons per symbol interval is simply

$$\mathsf{m}_\ell = s_\ell/e.$$

(e) The power penalty compared to an on-off-keyed intensity-modulated system operating at the same data rate.

Solution
The energy efficiency of multilevel intensity modulation is given in (10.5.10) and is repeated here

$$\mathcal{E} = \frac{d_{\text{min}}/2}{(L-1)d_{\text{min}}/2} = \frac{1}{L-1}.$$

For $L = 4$, this is a factor of one-third, or about -4.8 dB, compared to binary on-off keying.

10.9 *The effect of the extinction ratio on the optimal threshold*

Consider a single carrier system that transmits a mean power P at a symbol rate R. The length of the span is L km, and has an attenuation of κ dB/km. The receiver is an ideal photon-counting receiver ($\eta = 1$).

(a) Determine the expected number of photons for a mark E_1 and the expected number of photon for a space E_0 in terms of the expected number of photoelectrons per bit $\mathsf{W_b}$ and the transmitter extinction ratio e_x defined in (7.5.6).

Solution

For ideal photodetection the expected number of photons E is equal to the expected number of photocounts W. The mean number of photocounts per bit is $\mathsf{W_b} = (\mathsf{W}_1 + \mathsf{W}_0)/2$ and $e_x = \mathsf{W}_0/\mathsf{W}_1$. Therefore $\mathsf{W}_1 = 2\mathsf{W_b}/(1+e_x)$ and $\mathsf{W}_0 = 2\mathsf{W_b}e_x/(1+e_x)$.

(b) Derive an expression that relates the extinction ratio e_x to the error-rate p_e.

Solution

The solution requires the expression for the error rate in terms of $\mathsf{W_b}$ and ratio e_x. The expression for the $\overline{\gamma}^2 = \mathcal{Q}$-factor is given by

$$\overline{\gamma}^2 = \sqrt{\mathsf{W}_1} - \sqrt{\mathsf{W}_0} = \sqrt{\frac{2\mathsf{W_b}}{1+e_x}}\left(1-\sqrt{e_x}\right).$$

(c) Now suppose that the dark current in the photodetector is 10% of the mean photodetected electrical signal. Determine the modified extinction ratio required to achieve the same probability of error as in part (b).

Solution

The presence of dark current modifies the quantities as follows:

$$\begin{aligned}
\mathsf{W}_1' &= \mathsf{W}_1 + \mathsf{W}_{\mathsf{dark}} \\
\mathsf{W}_0' &= \mathsf{W}_0 + \mathsf{W}_{\mathsf{dark}} \\
e_x' &= \frac{\mathsf{W}_0'}{\mathsf{W}_1'} = \frac{\mathsf{W}_0 + \mathsf{W}_{\mathsf{dark}}}{\mathsf{W}_1 + \mathsf{W}_{\mathsf{dark}}} \\
&= \frac{2e_x\mathsf{W_b} + (1+e_x)\mathsf{W}_{\mathsf{dark}}}{2\mathsf{W_b} + (1+e_x)\mathsf{W}_{\mathsf{dark}}} = \frac{2e_x + 0.1(1+e_x)}{2+0.1(1+e_x)} = \frac{2.1e_x + 0.1}{0.1e_x + 2.1} \qquad (29) \\
\mathsf{W_b}' &= \frac{\mathsf{W}_1' + \mathsf{W}_0'}{2} = \mathsf{W_b} - \mathsf{W}_{\mathsf{dark}} \\
&= 0.9\mathsf{W_b}
\end{aligned}$$

where $\mathsf{W}_{\mathsf{dark}} = 0.1\mathsf{W_b}$ has been used. Equating the expressions for $\overline{\gamma}^2$ derived in part (b) and cancelling common terms gives

$$\sqrt{\frac{1.8}{1+e'_x}}\left(1-\sqrt{e'_x}\right) \quad = \quad \sqrt{\frac{2}{1+e_{x_{\text{mod}}}}}\left(1-\sqrt{e_{x_{\text{mod}}}}\right),$$

where $e_{x_{\text{mod}}}$ is the modified extinction ratio required to have the same probability of detection error. Given that e'_x can be expressed in terms of the original extinction ratio e_x using (29), this expression relates the original extinction ratio and the modified extinction ratio $e_{x_{\text{mod}}}$.

10.11 *Photon noise*

Let the power density spectrum for the spontaneous emission N_{sp} expressed in terms of the expected number of photons have a value of two at a wavelength of 1550 nm.

(a) Determine the power density spectrum N_{sp} from spontaneous emission in dBm/Hz and evaluate the noise power over a bandwidth of 25 GHz.

Solution
The power density spectrum in dBm/Hz is

$$N_{\text{sp}} \quad = \quad \log_{10}\Big(\underbrace{2}_{\mathsf{N}_{\mathsf{sp}}}\times\frac{hc}{\lambda}\Big) \quad = \quad \left(2\times\frac{6.626\times10^{-34}\times3\times10^{8}}{1.55\times10^{-6}}\right) \quad = \quad -155.9 \text{ dBm/Hz.}$$

The noise power in a bandwidth of 25 GHz is then $-155.9+10\log_{10}(25\times10^{9}) = -55.9$ dBm.

(b) Compare the power density spectrum N_{sp} to the power density spectrum for thermal noise $N_0 = kT_0$ assuming $\mathcal{R}=1$, and $R=50\,\Omega$.

Solution
At a temperature of 290 K, the thermal noise power density spectrum in dBm is

$$N_{\text{th}} \quad = \quad 10\log_{10}\left(10^{3}kT_0\right) \quad = \quad -174 \text{ dBm/Hz}$$

so that an optical amplifier with an equivalent noise of two photons per mode has a power density spectrum that is about 20 dB larger than the thermal noise power density spectrum at room temperature. This is one reason why thermal noise can be often be neglected when a lightwave amplifier is used.

(c) Let the expected number of signal photons E_{b} for a bit also have the value of two. Determine the probability of a detection error p_e for both heterodyne and homodyne detection including shot noise and spontaneous emission noise for an ideal photodetector ($\eta = 1$).

Solution

For homodyne demodulation with ideal photodetection we have

$$\begin{aligned} p_e &= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{2\mathsf{E_b}}{2\mathsf{N_{sp}}+1}}\right) \\ &= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{2(2)}{2(2)+1}}\right) \quad = \quad 0.103. \end{aligned}$$

For heterodyne demodulation with ideal photodetection we have

$$\begin{aligned} p_e &= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{\mathsf{E_b}}{\mathsf{N_{sp}}+1}}\right) \\ &= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{2}{3}}\right) \quad = \quad 0.124. \end{aligned}$$

(d) Repeat the last question neglecting photon noise and determine the relative error in p_e when photon noise is neglected.

Solution

Neglecting shot noise gives

$$\begin{aligned} p_e &= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{\mathsf{E_b}}{\mathsf{N_{sp}}}}\right) \\ &= \frac{1}{2}\text{erfc}(1) \quad = \quad 0.0786 \end{aligned}$$

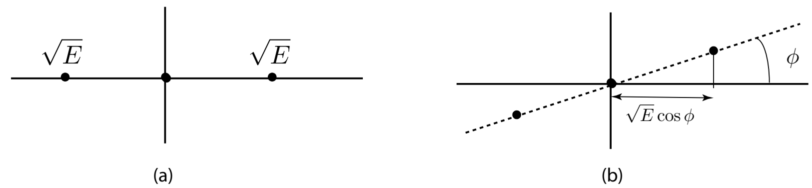
The relative error for each form of demodulation is

$$\begin{aligned} \text{Error(homo)} &= 100 \times \frac{0.103 - 0.0786}{0.103} \quad = \quad 23\%. \\ \text{Error(hetero)} &= 100 \times \frac{0.124 - 0.0786}{0.124} \quad = \quad 37\%, \end{aligned}$$

showing that the shot noise must be included for this example for an accurate calculation.

10.14 *Detection probabilities*

Consider a constellation that consists of three signal points as shown in figure below with one point at the origin.



(a) Referring to part (a) of the figure, suppose that the noise is additive with $E/N_0 = 5$ where E is the expected value for the symbol energy.

(i) Determine the optimal thresholds for detection.

Solution

Because the noise is additive, the thresholds are set halfway between the signal points with the two thresholds given as $\Theta_1 = -\sqrt{E}/2$ and $\Theta_2 = \sqrt{E}/2$.

(ii) Calculate the exact probability of a detection error.

Solution

The probability of a symbol error is given by (10.2.3) and repeated below

$$p_e = \frac{L-1}{L}\text{erfc}\left(\sqrt{\frac{3}{L^2-1}\frac{E}{N_0}}\right)$$

where the symbol energy E is equal to $E_b \log_2 L$. Substituting $L = 3$ and $E/N_0 = 5$ gives

$$p_e = \frac{2}{3}\text{erfc}\left(\sqrt{\frac{3}{8}(5)}\right) = 3.52 \times 10^{-2}.$$

(iii) Calculate the approximate probability of a detection error using the minimum distance.

Solution

The union bound in terms of d_{min} is given by (10.2.13) and repeated below

$$p_e \leq \frac{\overline{n}}{2}\text{erfc}\left(\sqrt{\frac{d_{\text{min}}^2}{4N_0}}\right),$$

For the three point signal constellation the average number of nearest neighbors $\overline{n}$ is $(1 +$

$2+1)/3 = 4/3$, $d_{\text{min}} = \sqrt{E}$ so that

$$\begin{aligned} p_e &\le \frac{2}{3}\text{erfc}\left(\sqrt{\frac{E}{4N_0}}\right) \\ &\le \frac{2}{3}\text{erfc}\left(\sqrt{\frac{5}{4}}\right) \quad = \quad 7.60 \times 10^{-2}, \end{aligned}$$

which overestimates the probability of a detection error by about a factor of two compared to the exact expression.

(b) Suppose that the demodulation occurs with a fixed phase error ϕ shown in part (b) of the figure. Repeat part (a) using the same thresholds.

Solution

With a phase error ϕ and the same thresholds, the amplitude of the signal along the x axis is reduced by $\sqrt{E}\cos\phi$ as is shown in part (b) of the figure. Therefore, the probability of a symbol error p_e is given by (10.2.3) and repeated here

$$p_e \quad = \quad \frac{2}{3}\text{erfc}\left(\sqrt{\frac{3}{8}(5)}\cos\phi\right).$$

The probability of a detection error using the union bound is

$$\begin{aligned} p_e &\le \frac{2}{3}\text{erfc}\left(\sqrt{\frac{E}{4N_0}}\right) \\ &\le \frac{2}{3}\text{erfc}\left(\sqrt{\frac{5}{4}}\cos\phi\right). \end{aligned}$$

10.19 *Phase-synchronous demodulation versus direct-photodetection demodulation in the presence of background noise*

This problem compares the probability of a detection error p_e of a shot-noise-limited phase-synchronous homodyne demodulation and photon counting in the presence of background noise.The background noise is modeled as a constant photogeneration arrival rate μ. For dark current, this term is $r_{\text{dark}} = e\mu_{\text{dark}}$ where μ_{dark} is a constant arrival rate and $\mathsf{W}_{\text{dark}} = \mu_{\text{dark}}T$ is the number of background photoelectrons in an interval T for a constant dark current arrival rate μ_{dark}.

(a) Using an appropriate wave-optics model for the background noise, derive an expression for the bit error rate for the phase-synchronous homodyne demodulation of binary phase-shift keying in the presence of background noise.

Solution
Including a background term generated from the dark current in the photodetector, the expression for E_p/N_0 for shot-noise-limited homodyne demodulation given in (8.2.16) is modified to read

$$\begin{aligned}\frac{E_p}{N_0} &= \frac{4ei_{\text{LO}}\mathsf{W}_\mathsf{p}}{\mathsf{W}_{\text{dark}} + 2ei_{\text{LO}}} \\ &= \frac{\mathsf{W}_\mathsf{p}}{\mathsf{W}_{\text{dark}}/\mathsf{W}_{\text{LO}} + 1/2}\end{aligned}$$

where W_b is the mean number of photocounts photons in a bit, W_{dark} is the number of noise counts, and $\mathsf{W}_{\text{LO}} = 4ei_{\text{LO}}$ is the mean number of photocounts generated from the local oscillator. This expression shows that the background noise term generated from dark current in the photodetector is reduced by the mixing gain.

(b) Compare this expression to the bit error rate for photon counting in the presence of a background noise term W_0 given in (9.5.40) when $\mathsf{W}_1 = 2\mathsf{W}_\mathsf{b}$ and $\mathsf{W}_0 = \mathsf{W}_{\text{dark}}$.

Solution
For photon counting with a background term, the optimal threshold is given in (9.5.38) and repeated here for $\mathsf{W}_1 = 2\mathsf{W}_\mathsf{b}$ and $\mathsf{W}_0 = \mathsf{W}_{\text{dark}}$

$$\Theta = \left\lfloor \frac{2\mathsf{W}_\mathsf{b} - \mathsf{W}_{\text{dark}}}{\log_e 2\mathsf{W}_\mathsf{b} - \log_e \mathsf{W}_{\text{dark}}} \right\rfloor .$$

The probability of a detection error is given in (9.5.40), which is repeated here for an equiprobable prior

$$p_e = \frac{1}{2} - \frac{1}{2}\sum_{\mathsf{m}=0}^{\Theta} \frac{1}{\mathsf{m}!}\left((\mathsf{W}_{\text{dark}})^{\mathsf{m}}\mathsf{e}^{-\mathsf{W}_{\text{dark}}} - 2(\mathsf{W}_\mathsf{b})^{\mathsf{m}}e^{-2\mathsf{W}_\mathsf{b}}\right) .$$

For this case, the effect of the background term is not scaled by the mixing gain (cf. Figure 9.15).

(c) Which modulation format is more robust to the presence of background noise?

Solution
Because of the mixing gain, phase synchronous detection is more tolerant of background noise generated from dark current in the photodetector compared to direct photodetection.

10.21 *Nearest neighbors for quadrature amplitude modulation*

The interior points, the exterior points, and the corner points of a square quadrature amplitude modulation constellation have a different numbers of nearest neighbors. Accounting for these differences, show that the mean number of nearest neighbors $\bar{n}$ for QAM is $4\left(1-1/\sqrt{L}\right)$.

Solution

The four corner points have two nearest neighbors. Excluding the four corner points, the four "edges" of length $(\sqrt{L}-2)$ have three two nearest neighbors. The remaining $(\sqrt{L}-2)^2$ interior point have four nearest neighbors so that

$$
\begin{aligned}
\bar{n} &= \tfrac{1}{L}\big(4(2)+4(\sqrt{L}-2)(3)+(\sqrt{L}-2)^2(4)\big) \\
&= \tfrac{4}{L}\big(2+3\sqrt{L}-6+L-4\sqrt{L}+4\big) \\
&= 4\left(1-1/\sqrt{L}\right).
\end{aligned}
$$

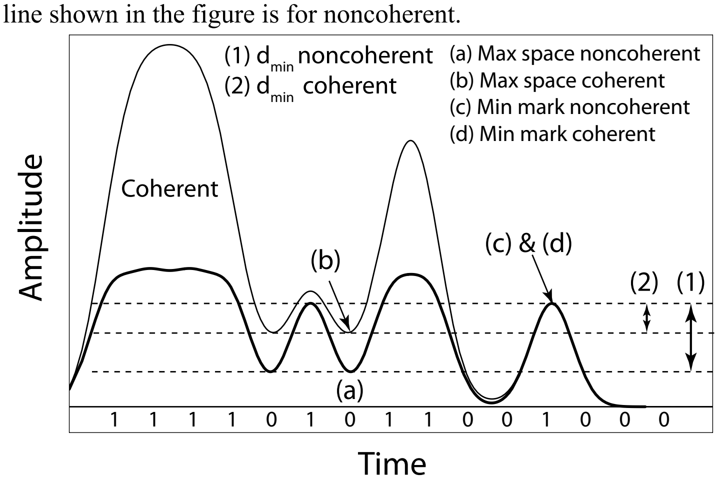
Chapter 11 Selected Solutions

11.1 *Minimum distance for coherent and noncoherent carriers*

(a) On a sketch or copy of Figure 11.14b, draw a line indicating the maximum amplitude for a space using a noncoherent carrier.
(b) Repeat for the maximum amplitude for a space using a coherent carrier.
(c) Repeat for the minimum amplitude for a mark using a coherent carrier.
(d) Repeat for the minimum amplitude for a mark using a noncoherent carrier. (This is the same value as for part (c).
(e) Using these values, determine which system has the largest minimum distance.

Solution

The heavy line shown in the figure is for noncoherent.



For this specific case, d_{min} for the noncoherent case is larger than d_{min} for the coherent case. For other patterns of pulses, this may not be true.

11.3 *Uncompensated intersymbol interference for intensity modulation*

For simple on-off keyed intensity modulation, the effect of intersymbol interference is to reduce the minimum sample value for a mark and increase the maximum sample value for a space thereby reducing the minimum separation d_{eye} compared to the minimum distance d_{min} in the absence of interference.

(a) The minimum high sample s_1' without noise occurs for an isolated mark because the neighboring spaces do not add to the value. Show that this worst-case value is

$$s_1' \quad = \quad s_1 - \epsilon \, \Delta s,$$

where $d_{10} = \Delta s = s_1 - s_0$ is the minimum distance in the absence of intersymbol interference, and

$$\epsilon = 1 - \frac{1}{S}\int_0^T s(t)\mathrm{d}t$$

is the proportion ϵ of the sample value for a mark that is lost because the pulse has spread to other symbol intervals, where $S = \int_{-\infty}^{\infty} s(t)\mathrm{d}t$ is the total signal in one pulse.

Solution
For a mark surrounded by spaces the worst case value s_1' is reduced by the proportion ϵ of the pulse that is lost to other signaling intervals as given above.

(b) Show that the maximum value s_0' for a space is

$$s_0' = s_0 + \epsilon\,\Delta s.$$

Solution
The same line of reasoning holds for a space. In the worst case, the proportion of the pulse ϵ that is lost is added to the space as given above.

(c) Show that the minimum separation $d_{\text{eye}} \doteq s_1' - s_0'$ in the presence of intersymbol interference is

$$d_{\text{eye}} = d_{10}(1 - 2\epsilon).$$

Solution
Subtracting the two expressions gives the separation d_{eye} between the two worst-case values as

$$\begin{aligned} d_{\text{eye}} &= s_1' - s_0' \\ &= s_1 - \epsilon\,\Delta s - s_0 - \epsilon\,\Delta s \\ &= d_{10}(1 - 2\epsilon). \end{aligned}$$

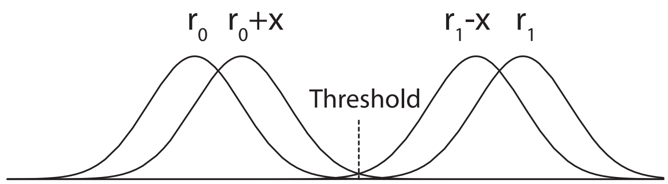
(d) Using $d_{10} = 2$, compare the minimum separation for intensity modulation to the minimum separation for binary phase-shift keying $d_{\text{min}} = 2\sum_{j\neq k}|s_{k-j}q_j|$ given in (11.1.6). Comment on the result. In what way is the interference similar? In what way is the interference different?

Solution

Using $d_{10} = 2$, the expression for intensity modulation is $d_{\text{min}} = 2(1 - 2\epsilon)$ which shows that the term $\sum_{j \neq k} |s_{k-j} q_j|$ for BPSK is replaced by the term $(1-2\epsilon)$ for intensity modulation. These expressions are similar because an isolated amplitude value of $+1$ surrounding by neighboring symbols of -1 has the same form as a mark value of two surrounded by neighboring space symbols of value zero. However, the expressions are not directly comparable because the values of BPSK are $s \in \{-1, 1\}$ whereas for intensity modulation, $s \in \{0, 2\}$. This means that neighboring BPSK symbols may add or subtract to any given symbol

11.6 *Worst-case intersymbol interference*

Consider the illustration shown below



for which the probability density functions for a mark and a space can be resolved into four separate probability density functions. Each of the four probability density functions is a gaussian distribution with the same variance σ^2. The probability density functions with means r_1 and r_0 represent symbols of the sequence that are not affected by intersymbol interference. The two other probability density functions with means $r_1 - x$ and $r_0 + x$ represent symbols of the sequence that have significant intersymbol interference. The threshold is set at $(r_1 + r_0)/2$.

(a) Determine the relationship between x and the intersymbol interference parameter ϵ defined in Problem 3.

Solution

The expressions for the mean values including the effect of ISI were derived for the solution to Problem 3 and are

$$
\begin{aligned}
r_1' &= r_1 - \epsilon\,\Delta r \\
r_0' &= r_0 + \epsilon\,\Delta r,
\end{aligned}
$$

where $\Delta r = r_1 - r_0$, and $x = \epsilon\,\Delta r$.

(b) Determine the conditional error probability $p_{1|0}$ using only the probability density function for a mark with an expected value r_1 and the probability density function for a space with an expected value of $r_0 + x$. Repeat for the conditional probability $p_{0|1}$ using r_0 and r_1.

Solution

The expression for $p_{1|0}$ is

$$p_{1|0} \approx \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\Theta}^{\infty} e^{-(y-r_0)^2/2\sigma_0^2} \mathrm{d}y,$$

where $r_0 \to r_0 + x$ because a shifted space distribution is used with the same threshold $\Theta = (r_1 + r_0)/2$. Then substitute $r' = \frac{r-(r_0+x)}{\sigma}$, $\mathrm{d}r' = \mathrm{d}r/\sigma$ and change the lower limit to $\mathrm{d}r' = \frac{\Theta-(r_0+x)}{\sigma} = \frac{(r_1+r_0)/2-r_0-x}{\sigma} = \mathcal{Q} - x/\sigma$. Therefore

$$p_{1|0} \approx \tfrac{1}{2}\mathrm{erfc}\big(\mathcal{Q} - x/\sqrt{2}\sigma\big).$$

The probability $p_{0|1}$ is $\frac{1}{2}\mathrm{erfc}(\mathcal{Q}/\sqrt{2})$ because the mean value for the mark distribution has not changed. Therefore, the total probability of error for an equiprobable prior is

$$p_e = \frac{1}{4}\mathrm{erfc}\big(\mathcal{Q} - x/\sqrt{2}\sigma\big) + \frac{1}{4}\mathrm{erfc}(\mathcal{Q}/\sqrt{2}).$$

(c) Repeat part (b) using the probability density function for a mark with an expected value of $r_1 - x$ and the probability density function for a space with an expected value of r_0.

Solution

Using symmetry arguments, the expression for $p_{0|1}$ for the shifted mark distribution is the same as part (b).

(d) Using the two conditional probability density functions from part (b) and the two conditional probability density functions from part (c), determine the probability of a detection error p_e when the priors are equal.

Solution

There are two mark distributions and two space distributions and thus there are four distributions in total. We have calculated three of them. The fourth is symmetric about the threshold with a mean mark value of $r_1 - x$ and a mean space value of $r_0 + x$ so that the error from this term is $\frac{1}{2}\mathrm{erfc}\big(\mathcal{Q} - 2x/\sqrt{2}\sigma\big)$. The total probability p_e of an error is

$$p_e \;=\; \tfrac{1}{4}\mathrm{erfc}(\mathcal{Q}/\sqrt{2}) + \tfrac{1}{4}\mathrm{erfc}\big(\mathcal{Q} - x/\sqrt{2}\sigma\big).$$

(e) Let $\Delta r = r_1 - r_0 = 10$ and $\sigma = 1$. Find the total probability of a detection error when $x = 1$ and determine the relative contribution from each of the four error terms—two from part (b) and two from part (c). Which term has the largest contribution? Why?

Solution

Using the values $\mathcal{Q} = 5$, and $\mathcal{Q} - x/\sigma = 4$. The probability of an error for the first term is $0.25\text{erfc}(5/\sqrt{2}) = 1.43 \times 10^{-7}$. The probability of an error for the second term is $0.25\text{erfc}(4/\sqrt{2}) = 1.59 \times 10^{-5}$. This results show that part of composite distribution that is "closest" dominate the total error calculation (>99% for this case).

(f) Using the results of part (e) and (a), compare this result with the probability of a detection error derived using (11.9.2) and thus comment on the conditions for which the minimum separation d_{eye} can be used to accurately determine the effect of the intersymbol interference.

Solution

Using the given values and the expression from part (a), the ISI parameter $\epsilon = x/\Delta r = 1/10$. Therefore, $p_e = \frac{1}{2}\text{erfc}\big(\mathcal{Q}(1-2\epsilon)/\sqrt{2}\big) = \frac{1}{2}\text{erfc}(4/\sqrt{2}) = 3.17 \times 10^{-5}$. Comparing this value to the "exact" expression from part (e), shows that using ϵ to estimate the error overestimates the error by about a factor of two for the parameters used in this problem. This overestimate is because ϵ is based on a worst-case pattern of a mark surrounded by spaces and a space surrounding by marks.

11.7 *Effect of group-velocity dispersion and laser linewidth on the intersymbol interference*

A system transmits R bits/second over a span of L km. A mark is represented by a gaussian pulse with a root-mean-squared timewidth T_{rms}. The modulated lightwave pulse has a root-mean-squared spectral width σ_λ. The intensity modulator has an extinction ratio $e_x = \mathsf{E}_0/\mathsf{E}_1$ (cf. (7.5.6)). The expected number of photons per bit is E_b. There are no other noise sources. Derive an expression for p_e in terms of E_b, e_x, and the intersymbol interference parameter ϵ defined in Problem 3.

Solution

In order to determine the ISI power penalty, we need an expression for E_1 in terms of E_b.

$$\mathsf{E}_1 = \frac{2\mathsf{E}_\mathsf{b}}{1+e_x} \tag{30a}$$

$$\mathsf{E}_0 = \frac{2\mathsf{E}_\mathsf{b} e_x}{1+e_x}. \tag{30b}$$

The expression for the intersymbol interference parameter ϵ defined in Problem 3 is

$$\begin{aligned}\epsilon &= 1 - \frac{1}{S}\int_{-1/2R}^{1/2R} e^{-t^2/2T_{\text{rms}}^2} \\ &= \text{erfc}(RT_{\text{rms}}/2\sqrt{2}).\end{aligned}$$

The value of E_1 in the presence of ISI is $\mathsf{E}_{1_{\text{ISI}}} = \mathsf{E}_1 - \epsilon\mathsf{E}_\mathsf{b}$ and $\mathsf{E}_{0_{\text{ISI}}} = \mathsf{E}_0 + \epsilon\mathsf{E}_\mathsf{b}$ so that

$$\mathsf{E}_{1_{\text{ISI}}} = \mathsf{E}_\mathsf{b}\left(\frac{2}{1+e_x} - \epsilon\right) \tag{31a}$$

$$\mathsf{E}_{0_{\text{ISI}}} = \mathsf{E}_\mathsf{b}\left(\frac{2e_x}{1+e_x} + \epsilon\right), \tag{31b}$$

where (30) has been used. Using the gaussian approximation given in (9.6.10), the probability p_e of a detection error when only signal-dependence noise is considered is

$$p_e = \sqrt{\mathsf{E}_{1\text{ISI}}} - \sqrt{\mathsf{E}_{0\text{ISI}}},$$

where the terms $\mathsf{E}_{1_{\text{ISI}}}$ and $\mathsf{E}_{0_{\text{ISI}}}$ are expressed in terms of E_b, e_x, and ϵ in (31).

11.8 *Minimum distance for coherent and noncoherent carriers*

Referring to Figure 11.14, let $r_p(t) = e^{-t^2/2T_{\text{rms}}^2}$.

(a) Assuming that only the adjacent symbols contribute to the interference, determine the minimum electrical amplitude of an isolated mark surrounded by spaces using the following methods:

(i) Adding the amplitude of pulses and then squaring the resulting amplitude of the sequence of pulses. This method is appropriate for a coherent carrier.
(ii) Squaring the amplitude of each isolated pulse and then adding the sequence of squared pulses.This method is appropriate for a noncoherent carrier.

(b) Repeat part (a) for an isolated space is surrounded by marks.

Solution
For the isolated mark, the value for coherent and noncoherent is the same and is equal to one. For an isolated space for coherent detection we sum and then square two shifted marks and then evaluate at $t = 0$

$$r(0) = \left(e^{-(t+T)^2/2T_{\text{rms}}^2} + e^{-(t-T)^2/2T_{\text{rms}}^2}\right)^2\Big|_{t=0} = 4e^{-T^2/T_{\text{rms}}^2}.$$

For a noncoherent system we square and then add so that

$$r(0) = \left(e^{-(t+T)^2/2T_{\text{rms}}^2}\right)^2 + \left(e^{-(t-T)^2/2T_{\text{rms}}^2}\right)^2\Big|_{t=0} = 2e^{-T^2/T_{\text{rms}}^2}.$$

(c) Using the results of the previous two parts, derive an expression for the minimum separation d_{eye} for both a coherent carrier and a noncoherent carrier. Comment on the result with respect to Figure 11.14.

Solution

For the coherent carrier, we have

$$d_{\text{eye}} = 1 - 4e^{-T^2/T_{\text{rms}}^2}.$$

For the noncoherent carrier we have

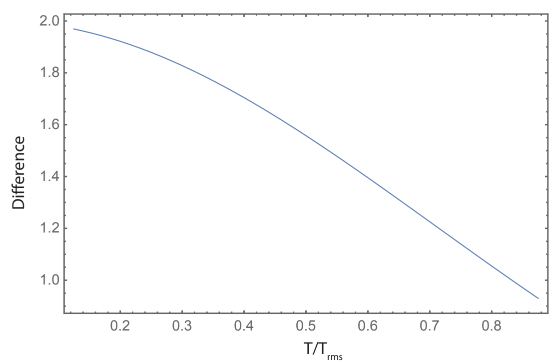
$$d_{\text{eye}} = 1 - 2e^{-T^2/T_{\text{rms}}^2}.$$

For this specific example the minimum separation is smaller for the coherent carrier because the interference from the adjacent symbol intervals adds constructively.

(d) Form the difference between the minimum separation derived using a coherent carrier and the minimum separation derived using a noncoherent carrier. Plot this difference over the range of $T/8 < T_{\text{rms}} < 7T/8$. Comment on the result.

Solution

The difference between the separations is $2e^{-T^2/T_{\text{rms}}^2}$. A plot over the range $1/8 < T/T_{\text{rms}} < 7/8$ is shown below



The difference decreases as the symbol interval T becomes large with respect to the pulse width T_{rms} because there is less interference between the pulses.

(e) Describe how you could estimate the minimum separation d_{eye} for a partially coherent carrier. What additional information is needed?

Solution

The degree of coherence of the source is required to estimate the minimum separation.

11.9 *Shot Noise Error*

Using Campbell's theorem, determine the contribution $\sigma^2_{\text{shot}}(kT)$ to the variance of the sample caused by the shot noise for the kth symbol interval. Note that when shot noise is present in an intensity-modulated waveform, the time-varying mean $r(t)$ used to evaluate the shot noise is given by

$$r(t) \quad = \quad i_0 + \langle \underline{s} \rangle \sum_{j=-\infty}^{\infty} p(t - jT),$$

where $\langle \underline{s} \rangle$ is the expected value of a stationary datastream of intensity-modulated symbols, $p(t)$ is the received pulse, and i_0 is a constant characterizing a background term.

Solution

Substituting the expression for the time-varying mean into the expression for the variance given in Campbell's theorem (cf. (6.7.18b)), the variance is

$$\begin{aligned}
\sigma^2_{\text{shot}}(kT) &= \int_{-\infty}^{\infty} r(kT - t) f^2(t) \mathrm{d}t \Big|_{t=kT} \\
&= \int_{-\infty}^{\infty} \Big(i_0 + \langle \underline{s} \rangle \sum_{j=-\infty}^{\infty} p(t - jT) \Big) f^2(t) \mathrm{d}t \Big|_{t=kT},
\end{aligned}$$

where $f(t)$ is the impulse response of the detection filter.

Chapter 12 Selected Solutions

12.1 *Second-order phase-locked loops*

A loop filter used in a second-order phase-locked loop has the transfer function

$$H_L(f) = 1 + \frac{a}{\mathrm{i}2\pi f}$$

where a is a constant. The phase-locked loop response under a linear approximation is described by

$$Z(f) = C_1 H_L(f)\big(\Phi(f) - \widehat{\Phi}(f)\big)$$

where $\Phi(f)$ is the Fourier transform of the phase and $\widehat{\Phi}(f)$ is the Fourier transform of the output of the controlled oscillator.

(a) Starting with (12.2.12), and using the properties of the Fourier transform, show that

$$\widehat{\Phi}(f) = \frac{C_2}{\mathrm{i}2\pi f} Z(f).$$

Solution
Taking the Fourier transform of (12.2.12), which is repeated here

$$\frac{\mathrm{d}\widehat{\phi}(t)}{\mathrm{d}t} = C_2 z(\tau),$$

and solving for $Z(f)$ gives

$$Z(f) = C_1 H_L(f)\big(\Phi(f) - \widehat{\Phi}(f)\big) \tag{32}$$

(b) Define $H(f) = \widehat{\Phi}(f)/\Phi(f)$ as the ratio of the phase estimate $\widehat{\Phi}(f)$ to the input phase $\Phi(f)$. Show that $H(f)$ can be written in the form

$$H(f) = \frac{2\mathrm{i}\,\zeta\,(f/f_n) + 1}{-(f/f_n)^2 + 2\,\mathrm{i}\zeta\,(f/f_n) + 1}. \tag{33}$$

Express the natural frequency f_n and the damping parameter ζ in terms of the parameters a, C_1, and C_2.

Solution
Let $A = C_2/\mathrm{i}2\pi f$ and $B = C_1(1 + a/\mathrm{i}2\pi f)$. Then substituting $Z(f)$ into (32) gives

$$\widehat{\Phi}(f) = AB\big(\Phi(f) - \widehat{\Phi}(f)\big).$$

Solving for the ratio $H(f) = \widehat{\Phi}(f)/\Phi(f)$ gives

$$
\begin{aligned}
\frac{\widehat{\Phi}(f)}{\Phi(f)} &= \frac{AB}{AB+1} \\
&= \frac{C_1C_2(a+2i\pi f)}{-4\pi^2 f^2 + C_1C_2(a+2i\pi f)} \\
&= \frac{2i\pi f/a+1}{-4\pi^2 f^2/aC_1C_2 + 2i\pi f/a + 1}
\end{aligned}
$$

Equating terms with (33) gives $f_n = \sqrt{aC_1C_2}$ and $\zeta = \sqrt{aC_1C_2}/2a$.

(c) The error transfer function $H_e(f)$ is defined as the difference between the ideal transfer function $H(f) = 1$ and the actual transfer function $H_L(f)$ so that $H_e(f) = 1 - H_L(f)$. Derive $H_e(f)$ and show, in the absence of noise, that $\widehat{\phi} = \phi$ in steady state.

Solution

The error transfer function $H_e(f)$ is

$$
H_e(f) \quad = \quad 1 - H_L(f) \quad = \quad -\frac{a}{\mathrm{i}2\pi f}
$$

The corresponding impulse response for the loop error is a simply a derivative and goes to zero in the steady state.

12.2 *Nonlinear analysis of a phase-locked loop (requires numerics)*

A phase-locked loop that is not well-locked does not satisfy $\sin\theta_e \approx \theta_e$. The loop response is nonlinear, and the phase-noise probability density function is no longer gaussian. For a first-order phase-locked loop, the phase-noise probability density function is[2]

$$
f(\theta_e) = \frac{e^{\alpha\cos\theta_e}}{2\pi I_0(\alpha)},
$$

where $\alpha = 1/\sigma^2_{\theta_e}$ and $I_0(x)$ is the modified Bessel function of the first kind and order zero. Suppose that this probability density function is approximated by a zero-mean gaussian distribution $f(\theta_e)$ characterized by a root-mean-squared phase error σ_{θ_e} expressed in terms of degrees. Determine σ_{θ_e} such that the squared error

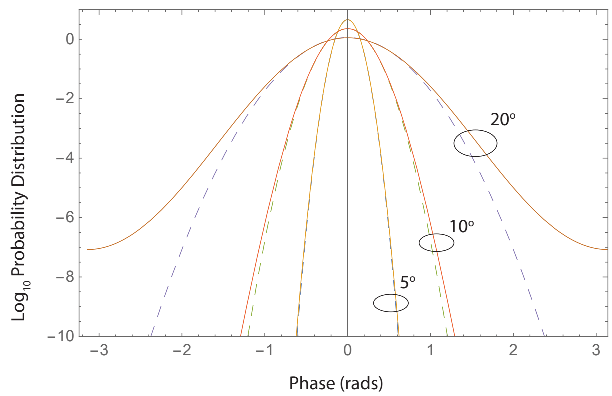
$$
\sqrt{\int_{-\pi}^{\pi} |f_{\text{exact}}(\theta_e) - f_{\text{gauss}}(\theta_e)|^2 \, \mathrm{d}\theta_e}, \tag{34}
$$

[2] See Tikhonov, V. I., *Phase-lock automatic frequency control application in the presence of noise*, Automatika i Telemekhanika, 21(3):209–14, 1960.

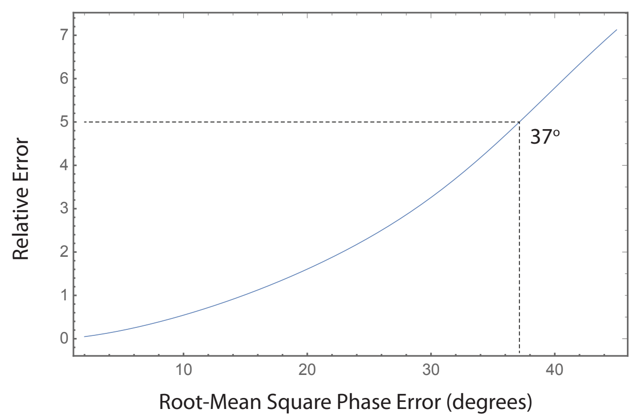
is less than five percent. Determine the approximate range of the validity of using a gaussian distribution for the probability density function of the phase noise.

Solution

A plot of (34) along with a gaussian approximation is shown in the figure below for root-mean squared phase errors of 5^o, 10^o, and 20^o.



The relative error using (34) is shown below



The relative error is less than 5% for a root-mean squared error of 37^o.

12.7 *Comparing polarization demodulation to I/Q demodulation*

Suppose that the estimated polarization basis is misaligned so that the block sample value **r** after the misalignment is related to the block input **s** at the transmitter by

$$\mathbf{r} = \mathbb{T}\mathbf{s},$$

where $\mathbb{T}$ is the polarization transformation given in (2.3.59) with $\chi = 0$ so that the misalignment is described by only the angle ξ. Compare the functional form of this kind of misalignment to the effect of a constant phase error θ_e in the estimate in the I-Q axes for the demodulation of the two quadrature signal components.

Solution

The general lossless polarization transformation $\mathbb{R}(\xi, \chi)$ is given by (2.3.59), which is repeated below

$$\mathbb{R}(\xi, \chi) \quad = \quad [\mathbf{J}_1, \mathbf{J}_2] \quad = \quad \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}.$$

where $a = \cos\xi\cos\chi - \mathrm{i}\sin\xi\sin\chi$, and $b = -\sin\xi\cos\chi + \mathrm{i}\cos\xi\sin\chi$. For $\chi = 0$, $a = \cos\xi$ and $b = -\sin\xi$ so that

$$\mathbb{R}(\xi, \chi = 0) \quad = \quad \begin{bmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{bmatrix}.$$

This matrix is identical to the matrix used for an I-Q rotation given in (12.2.7). When $\chi \neq 0$, this correspondance is not valid because the (lossless) polarization misalignment must then be described on the surface of the Poincaré sphere and not on a circle (or the equator of the Poincaré sphere).

12.12 *Polarization control*

Let the received lightwave with bit energy E_b be linearly polarized along a direction defined by the unit vector $\widehat{\mathbf{p}}_{\mathbf{c}}$. Let the local oscillator be linearly polarized along a direction defined by the unit vector $\widehat{\mathbf{p}}_{\mathbf{lo}}$. Let the loc and $\widehat{\mathbf{p}}_{\mathbf{lo}}$ be the unit vectors for a linearly-polarized carrier and the local oscillator, respectively.

(a) Derive an expression for the demodulated bit energy $E_b(\theta)$ in terms of the angle θ between $\widehat{\mathbf{p}}_{\mathbf{c}}$ and $\widehat{\mathbf{p}}_{\mathbf{lo}}$.

Solution

The energy is

$$\begin{aligned} E_b(\theta) &= \int_T |r(t)\widehat{\mathbf{p}}_{\mathbf{c}} \cdot \widehat{\mathbf{p}}_{\mathbf{lo}}|^2 \mathrm{d}t \\ &= E_b \cos^2\theta. \end{aligned}$$

(b) Suppose that a polarization estimator can track the angle so that the probability density function of $\underline{\theta}$ after estimation is a zero-mean gaussian with a variance σ_θ^2. Determine the maximum variance allowed for the estimator to limit the power penalty in the received signal to <1 dB for 99% of the cases.

Solution

The solution requires the probability density function of $\cos^2 \underline{\theta}$ where $\underline{\theta}$ is a zero-mean gaussian random variable. For small errors, an estimate of the probability density function can be obtained by expanding $\cos^2 \underline{\theta}$ in a power series so that

$$\cos^2 \underline{\theta} \approx 1 - \underline{\theta}^2.$$

Using 1 dB = 0.794, to have a 1 dB power penalty greater than 99

$$\int_0^{1/5} p_{\underline{\theta^2}}(\theta^2) \mathrm{d}\theta^2 \geq 0.99,$$

where the upper limit of $1/5 \approx 1 - 0.794$ is the value of θ^2 that produces a power penalty of 1 dB. When $\underline{\theta}$ is a zero-mean *real* gaussian random variable, $\underline{\theta}^2$ is a central chi-square random variable with one degree of freedom. The probability density function is given by (2.2.37) and is repeated here

$$f(z) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} z^{-1/2} e^{-z/2\sigma^2},$$

where $z = \theta^2$ and σ is the variance of the zero-mean gaussian. Evaluating the integral

$$\int_0^{0.2} \frac{1}{\sqrt{2\pi\sigma^2}} z^{-1/2} e^{-z/2\sigma^2} \mathrm{d}z$$

gives erf$(1/\sqrt{10}\sigma)$. Setting this term equal to 0.99, the maximum variance in the estimator is given as $\sigma = 0.173$. This variance will depend on the signal-to-noise ratio over the time interval used for the estimation.

Chapter 13 Selected Solutions

13.4. *Relationship between euclidean distance and Hamming distance*

(a) Show that for codebit energy E_c equal to 1, the relationship between the minimum Hamming distance d_{min} and the minimum euclidean distance d_{min} for binary phase-shift keying is given by

$$d_{\text{min}} = 2\mathsf{d}_{\text{min}}.$$

Solution
For BPSK, each codebit in the codeword is mapped to one of two antipodal values $\pm\sqrt{E_c}$. For E_c equal to 1, this is the bipolar alphabet $\{-1, 1\}$. For each component at which the two codewords differ, the corresponding components of the BPSK signal are separated by the single-letter euclidean distance $2\sqrt{E_c} = 2$. Multiplying the minimum Hamming distance d_{min} by the single-letter euclidean distance 2 gives the minimum euclidean distance d_{min} for the codeword as

$$d_{\text{min}} = 2\mathsf{d}_{\text{min}}.$$

(b) The expression relating the squared euclidean distance d^2_{min} and the Hamming distance d_{min} is given by (13.2.20). For E_c equal to one it is

$$d^2_{\text{min}} \quad = \quad 4\mathsf{d}_{\text{min}}.$$

The left side of the second equation is the square of the left side of the first equation However, the right side of the second equation is not the square of the right side of the first equation. Why?

Solution
The Hamming distance describes the difference between codewords in terms of the number of components that are different. This metric is not defined in terms of the euclidean distance or to the euclidean distance squared, which is related to the energy in each symbol. Therefore, the same Hamming distance scales both the single-letter euclidean distance and the single-letter euclidean distance squared. For this reason, the right side of the second equation is not the square of the right side of the first equation

13.6 *Coding gain for a repetition code*

(a) Determine the probability of a block error p_e for an uncoded sequence of three bits each with an energy E_b and independent bit error ρ.

Solution
The probability of a correct detection event is repeated here

$$p_c(\text{uncoded}) \quad = \quad (1 - \rho(\text{uncoded}))^n.$$

For BPSK, $\rho(\text{uncoded}) = \frac{1}{2}\text{erfc}\sqrt{E_b/N_0}$ and $n = 3$ so that

$$p_c(\text{uncoded}) \quad = \quad \left(1 - \tfrac{1}{2}\text{erfc}\sqrt{E_b/N_0}\right)^3.$$

(b) Determine the probability of a block error p_e for hard-decision decoding using a $(3, 1, 3)$ repetition code.

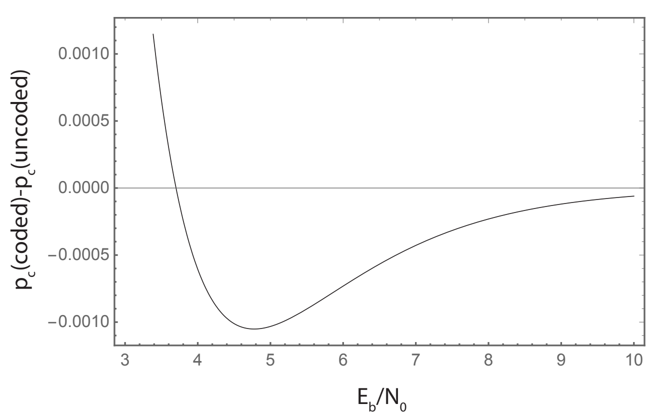
Solution
The $(3, 1, 3)$ repetition code can correct one error so that $\mathsf{t} = 1$. Therefore, probability of a correct detection event is

$$\begin{aligned} p_c(\text{coded}) \quad &= \quad \sum_{\ell=0}^{1} \binom{n}{\ell} \rho(\text{coded})^\ell \, (1 - \rho(\text{coded}))^{n-\ell} \\ &= \quad (1 - \rho(\text{coded}))^3 + 3\rho(\text{coded})\,(1 - \rho(\text{coded}))^2\,, \end{aligned}$$

where $\binom{3}{0} = 1$ and $\binom{3}{1} = 3$ have been used and $\rho(\text{coded}) = \frac{1}{2}\text{erfc}\sqrt{E_b/3N_0}$ with $E_c = RE_b = E_b/3$.

(c) Determine the value of E_b/N_0 for which the probability of a block error for an uncoded block is equal the probability of a block error for a coded block.

Solution
A plot of $p_c(\text{coded}) - p_c(\text{uncoded})$ is shown below.



The two methods produce the same probability when E_b/N_0 is about 3.7.

(d) For values of E_b/N_0 larger than the value determined in part (c), is there a coding gain? Explain.

Solution
For values of E_b/N_0 greater than 3.7, the difference $p_c(\text{coded}) - p_c(\text{uncoded})$ is negative meaning that the probability of a correct detection event using the repetition code is less than the probability not using a code. This is a negative coding gain.

(e) Show that the hard-decision coding gain of any $(n, 1, n)$ code is negative for a large value of E_b/N_0. Explain why.

Solution
For large values of E_b/N_0, the probability of a block error for an uncoded block can be approximated by (cf. (13.2.15b))

$$p_e(\text{uncoded}) \quad \approx \quad n\rho(\text{uncoded}).$$

Using $\rho(\text{uncoded}) = \frac{1}{2}\text{erfc}\sqrt{E_b/N_0}$ and the approximation $\text{erfc}(x) \approx e^{-x^2}$ for large x gives

$$p_e(\text{uncoded}) \quad \approx \quad \frac{n}{2}e^{-E_b/N_0}.$$

For a coded block, p_e can be approximated by (13.2.19), which is repeated here

$$p_e(\text{coded}) \quad \approx \quad \frac{n_{t+1}}{2}e^{-(E_b/N_0)R_c\mathsf{d}_{\mathsf{min}}/2},$$

where $n_{t+1} \doteq \binom{n}{\mathsf{t}+1}$ is the number of error patterns with $\mathsf{t}+1$ errors. For $(n, 1, n)$ repetition with code rate $R_c = 1/n$ and $\mathsf{d}_{\mathsf{min}} = n$ the expression reads

$$p_e \quad \approx \quad \frac{n_{t+1}}{2}e^{-E_b/2N_0}.$$

Because $e^{-E_b/2N_0} > e^{-E_b/N_0}$, and $n_{t+1} > n$ for $n > 3$, $p_e(\text{uncoded}) > p_e(\text{uncoded})$ for any $(n, 1, n)$ code.

13.8 *Pareto random variable*

(a) Show that if $\underline{y}$ is a random variable with an exponential probability density function given by

$$f_{\underline{y}}(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases},$$

then the random variable $\underline{x} = e^{\underline{y}}$ has a Pareto probability density function given by

$$f_{\underline{x}}(x) = \begin{cases} \lambda x^{-(\lambda+1)} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

Solution

The transformation $\underline{x} = e^{\underline{y}}$ must preserve probabilities on intervals so that $f_{\underline{x}}(x)\mathrm{d}x = f_{\underline{y}}(y)\mathrm{d}y$. Using this expression, the transformed distribution is given by (see Problem 2.15)

$$f_{\underline{x}}(x) = f_{\underline{y}}\Big(T^{-1}(x)\Big)\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|,$$

where T is equal to an exponential function so that $T^{-1}(x) = \log x$ for $x \geq 1$ and 0 otherwise. Given $\log x = y$, $|\mathrm{d}y/\mathrm{d}x| = x^{-1}$. Substituting $T^{-1}(x) = \log x$ and $|\mathrm{d}y/\mathrm{d}x| = x^{-1}$ into the previous equation gives

$$\begin{aligned} f_{\underline{x}}(x) &= f_{\underline{y}}\Big(T^{-1}(x)\Big)\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| \\ &= \lambda e^{-\lambda \log x} x^{-1} \qquad x \geq 1 \\ &= \lambda x^{-(\lambda+1)} \qquad x \geq 1. \end{aligned}$$

(b) Derive the mean and the variance of the Pareto probability density function.

Solution

The mean is

$$\int_1^\infty x\lambda x^{-(\lambda+1)}\mathrm{d}x = \frac{\lambda x^{1-\lambda}}{1-\lambda}\Big|_1^\infty.$$

The upper limit is equal to zero when λ is greater than one. Otherwise it is undefined or is infinite. Therefore the mean is

$$\langle f_{\underline{x}}(x)\rangle = \frac{\lambda}{\lambda - 1} \qquad \text{for } \lambda > 1.$$

The mean-square value is

$$\int_1^{\infty} x^2 \lambda x^{-(\lambda+1)} \mathrm{d}x \quad = \quad \left.\frac{\lambda x^{2-\lambda}}{2-\lambda}\right|_1^{\infty}.$$

The upper limit is equal to zero when λ is greater than two. Otherwise it is undefined or is infinite. Therefore the mean square value is

$$\langle f_{\underline{x}}^2(x)\rangle \quad = \quad \frac{\lambda}{\lambda-2} \qquad \text{for } \lambda > 2.$$

The variance is

$$\begin{aligned} \sigma^2 &= \langle f_{\underline{x}}^2(x)\rangle - \langle f_{\underline{x}}(x)\rangle^2 \\ &= \frac{\lambda}{\lambda-2} - \frac{\lambda^2}{(\lambda-1)^2} \\ &= \frac{\lambda}{(\lambda-2)(\lambda-1)^2} \qquad \text{for } \lambda > 2. \end{aligned}$$

13.13 *The cutoff rate and capacity for phase-shift keying*

Using the large-signal approximation for the capacity of the phase-shift-keyed information channel given in (14.3.15), repeated here as

$$\mathsf{C} \quad \approx \quad \tfrac{1}{2}\log(4\pi E/eN_0), \tag{35}$$

and the large-argument expansion for the modified Bessel function $I_0(x)$ of the first kind of order zero, which is given by

$$I_0(x) \quad \approx \quad \frac{1}{\sqrt{2\pi x}} e^x,$$

do the following.

(a) Show that, for the same value of E/N_0, the offset in the rate between the curve for the capacity and the curve for the cutoff rate shown in Figure 13.9 approaches the constant value $\frac{1}{2}\log_2(4/e) \approx 0.28$.

Solution

The cutoff rate for phase shift keying is given in (13.4.7) and repeated here

$$R_0 \quad = \quad -\log\big(e^{-E/2N_0} I_0(E/2N_0)\big).$$

Substituting the approximation for $I_0(x)$ given in the problem gives

$$R_0 \quad \approx \quad -\log(\pi E/N_0)^{-1/2} \quad = \quad \tfrac{1}{2}\log(\pi E/N_0). \tag{36}$$

The difference between the capacity C and the cutoff rate R_0 in bits is then

$$\begin{aligned}\mathsf{C} - R_0 &\approx \tfrac{1}{2}\log_2(4\pi E/eN_0) - \tfrac{1}{2}\log_2(\pi E/N_0) \\ &= \frac{1}{2}\log_2\left(\frac{4\pi(E/N_0)}{e\pi(E/N_0)}\right) \\ &= \frac{1}{2}\log_2(4/e).\end{aligned}$$

(b) Show that for the same rate, the offset in E/N_0 between the curve for capacity and the curve for the cutoff rate approaches the constant value $4/e \approx 1.68$ dB.

Solution

Examining the expression for the capacity C given in (35) and the expression for the cutoff rate R_0 for a large value of E/N_0 given in (36), the argument of the logarithm function for the capacity C is a factor of $4/e$ larger compared to the argument of the logarithm function for the cutoff rate R_0. Therefore, for large values of E/N_0 and the same information rate, this leads to an offset in E/N_0 that is approximately $4/e \approx 1.68$ dB between C and R_0. This offset is shown in Figure 13.10.

Chapter 14 Selected Solutions

14.3 *Discrete capacity using an exponential probability density function*

The large-signal limit for the capacity of a Poisson channel can be approximated as

$$\begin{aligned} \mathsf{C} &= H(\underline{\mathsf{r}}) - H(\underline{\mathsf{r}}|\underline{\mathsf{s}}) \\ &\approx \tfrac{1}{2}\log \mathsf{E} \\ &\approx \tfrac{1}{2}\mathsf{C}_{\mathsf{w}}. \end{aligned}$$

This limit can be derived using a central chi-square probability density function with one degree of freedom. Show that when this probability density function is replaced by an exponential function for $p(\mathsf{s})$, the resulting capacity is smaller than $\frac{1}{2}\mathsf{C}_{\mathsf{w}}$ by the constant term $\frac{1}{2}(\log_e 2\pi - \gamma)$, where Euler's constant γ is 0.5772.

Solution

The information rate for an exponential prior with mean E is given by

$$\mathsf{R} = H(\underline{\mathsf{r}}) - H(\underline{\mathsf{r}}|\underline{\mathsf{s}}),$$

where the conditional entropy is based on an exponential prior.

To achieve capacity, the received entropy must be maximized. This requires a Gordon distribution at the receiver (cf. Section 14.2). For large signal values, the Gordon distribution can approximated by a continuous exponential distribution (cf. Section 6.3.4) with the received entropy $H(\underline{\mathsf{r}})$ given by

$$H(\underline{\mathsf{r}}) = 1 + \log \mathsf{E} = \log_e(e\mathsf{E}) \qquad \text{nats.}$$

The conditional entropy is (cf. (14.1.3))

$$H(\underline{\mathsf{r}}|\underline{\mathsf{s}}) = -\sum_{\mathsf{s}} p(\mathsf{s}) \sum_{\mathsf{r}} p(\mathsf{r}|\mathsf{s}) \log p(\mathsf{r}|\mathsf{s}),$$

For large signal levels, $p(\mathsf{r}|\mathsf{s})$ is well-approximated by a one-dimensional gaussian distribution with a variance σ^2 equal to the mean signal value s so that (cf. (14.3.1))

$$-\sum_{\mathsf{r}} p(\mathsf{r}|\mathsf{s}) \log p(\mathsf{r}|\mathsf{s}) \approx \tfrac{1}{2}\log(2\pi e \mathsf{s}).$$

Substituting this expression into $H(\underline{\mathsf{r}}|\underline{\mathsf{s}})$ gives

$$H(\underline{\mathsf{r}}|\underline{\mathsf{s}}) = -\frac{1}{2\mathsf{E}} \int_0^\infty e^{-s/\mathsf{E}} \log(2\pi e \mathsf{s}) d\mathsf{s},$$

where $e^{-s/\mathsf{E}}/\mathsf{E}$ is the exponentially-distributed prior. Let $x = s/\mathsf{E}$ so that $\mathrm{d}x = \mathrm{d}s/\mathsf{E}$. Using this change of variable, write $\log(2\pi e\mathsf{s}) = \log(2\pi e\mathsf{E}) + \log x$. Substituting these expressions and using $\log_e$ gives

$$H(\underline{\mathsf{r}}|\underline{\mathsf{s}}) \quad = \quad \frac{1}{2}\log_e(2\pi e\mathsf{E}) + \frac{1}{2}\int_0^\infty e^{-x}\log_e x\mathrm{d}x.$$

The integral evaluates to $-\gamma$ where γ is the Euler constant. Using this expression, the rate for an exponential prior is

$$\begin{aligned}
\mathsf{R} &= H(\underline{\mathsf{r}}) - H(\underline{\mathsf{r}}|\underline{\mathsf{s}}) \\
&= \log_e(e\mathsf{E}) - \tfrac{1}{2}\big(\log_e(e\mathsf{E}) + \log_e(2\pi) - \gamma\big) \\
&= \tfrac{1}{2}\mathsf{C}_\mathsf{w} - \tfrac{1}{2}\big(\log_e(2\pi) - \gamma\big).
\end{aligned}$$

This expression is slightly different than the one given in the problem statement.

14.5 *Entropy of a Poisson probability mass function*

(a) Show that the entropy of the Poisson probability distribution

$$
\begin{aligned}
H(\mathsf{E}) &= -\sum_{\mathsf{k}=0}^{\infty} p(\mathsf{k}) \log_e p(\mathsf{k}) \\
&= -\sum_{\mathsf{k}=0}^{\infty} \frac{\mathsf{E}^{\mathsf{k}} e^{-\mathsf{E}}}{\mathsf{k}!} \log_e \left(\frac{\mathsf{E}^{\mathsf{k}} e^{-\mathsf{E}}}{\mathsf{k}!} \right)
\end{aligned}
$$

can be written as

$$
H(\mathsf{E}) = \mathsf{E} - \mathsf{E} \log_e \mathsf{E} + e^{-\mathsf{E}} \sum_{\mathsf{k}=2}^{\infty} \frac{\mathsf{E}^{\mathsf{k}} \log_e \mathsf{k}!}{\mathsf{k}!} \qquad \text{nats.}
$$

Solution

$$
\begin{aligned}
H(\mathsf{E}) &= -\sum_{k=0}^{\infty} \frac{\mathsf{E}^{k} e^{-\mathsf{E}}}{k!} \log_e \left(\frac{\mathsf{E}^{k} e^{-\mathsf{E}}}{k!} \right) \\
&= -e^{-\mathsf{E}} \sum_{k=0}^{\infty} \frac{\mathsf{E}^{k}}{k!} \left(\log_e \frac{1}{k!} + k \log_e \mathsf{E} - \mathsf{E} \right) \\
&= \mathsf{E} \times e^{-\mathsf{E}} \underbrace{\sum_{k=0}^{\infty} \frac{\mathsf{E}^{k}}{k!}}_{e^{\mathsf{E}}} - \mathsf{E} \log_e \mathsf{E} e^{-\mathsf{E}} \underbrace{\sum_{k=1}^{\infty} \frac{\mathsf{E}^{(k-1)}}{(k-1)!}}_{e^{\mathsf{E}}} + e^{-\mathsf{E}} \sum_{k=0}^{\infty} \frac{\mathsf{E}^{k} \log_e k!}{k!} \\
&= \mathsf{E}\left(1 - \log_e(\mathsf{E})\right) + e^{-\mathsf{E}} \sum_{k=0}^{\infty} \frac{\mathsf{E}^{k} \log_e k!}{k!} \qquad \text{nats}
\end{aligned}
$$

where units of nats are convenient because the probability distribution is defined in terms of exponentials.

(b) Using this expression, derive an approximation for the entropy of a Poisson probability mass function for E much smaller than one.

Solution

For E much smaller than one, the first term in $H(\mathsf{E})$ dominates with

$$
H(\mathsf{E}) \approx \mathsf{E}\left(1 - \log_e \mathsf{E}\right),
$$

which is (14.2.8a). A further approximation may be obtained by neglecting one compared to $-\log_e \mathsf{E}$ leading to $H(\mathsf{E}) \approx -\mathsf{E} \log_e \mathsf{E}$, which is (14.2.8b).

14.9 *Binary capacity in a small-signal regime*

Consider the binary detection of a lightwave signal in a small-signal regime for which the expected number of photons $\mathsf{E_b}$ is much smaller than one.

(a) Derive the optimal threshold Θ in terms of $\mathsf{E_b}$ and the prior probability p_1.

Solution

The system has a mean number of signal photons $\mathsf{E_1} = \mathsf{E_b}/p_1$ for a mark. This signal is added to a background noise term N for a space. When there is no additive nise, this is a Z channel. For this case, the threshold Θ is equal to zero.

(b) Derive the terms required to form the mutual information as a function $\mathsf{E_b}$ and the prior probability p_1.

Solution

For a Z-channel with the threshold Θ set to zero, the four condition probabilities required to evaluate the mutual information are

$$\begin{aligned} p_{0|1} &= e^{-\mathsf{E_b}/p_1} \\ p_{1|1} &= 1 - p_{0|1} \\ p_{0|0} &= 1 \\ p_{1|0} &= 0 \end{aligned}$$

(c) Determine the capacity for $\mathsf{E_b} = 1$. (Requires numerical root finding.)

Solution

The mutual information for a binary channel is given in (14.2.14). Substituting the expressions for the conditional probabilities gives

$$\begin{aligned} I(\mathsf{E_b}, p_1) = & -p_1\left(1 - e^{\mathsf{E_b}/p_1}\right)\log p_1 \\ & -\left(1 - p_1(e^{\mathsf{E_b}/p_1})\right)\log\left(1 - p_1(e^{\mathsf{E_b}/p_1})\right) \\ & +p_1 e^{\mathsf{E_b}/p_1}\log e^{\mathsf{E_b}/p_1} \end{aligned}$$

This expression is plotted in Figure 14.6 as a function of p_1 for several values of $\mathsf{E_b}$. When $\mathsf{E_b} = 1$, the mutual information is maximized when $p_1 = 0.41$, and gives the capacity as $\mathsf{C} = 0.44$ bits.

(d) Expand the expression for the mutual information in a power series in $\mathsf{E_b}$ keeping only the first term.

Solution
Using the power series expansions $e^{-x} \approx 1 - x$ and $\log(1 - x) \approx -x$ gives

$$I(\mathsf{E_b}, p_1) \quad \approx \quad \mathsf{E_b}\left(-\frac{\mathsf{E_b}}{p_1} + \mathsf{E_b} - \log p_1\right). \tag{37}$$

(e) Using the term from part (d), determine the value of p_1 that maximizes the mutual information for a given value of $\mathsf{E_b}$.

Solution
Taking the derivative of the appropriate form of the mutual information $I(\mathsf{E_b}, p_1)$ derived in part (d) with respect to p_1 and setting the resulting expression equal to zero gives

$$\frac{\mathsf{E_b}^2}{p_1^2} - \frac{\mathsf{E_b}}{p_1} \quad = \quad 0.$$

This equation has a solution when $p_1 = \mathsf{E_b}$.

(f) For the optimal prior probability determined in part (e) and $\mathsf{E_b}$ much smaller than one, show that the expression for the mutual information reduces to

$$-\mathsf{E_b} \log_2 \mathsf{E_b} \qquad \text{bits},$$

which is the small-signal limit of the entropy of a Poisson probability distribution.

Solution
Substituting $p_1 = \mathsf{E_b}$ into (37) gives the channel capacity as

$$\mathsf{C} \quad \approx \quad \mathsf{E_b}(\mathsf{E_b} - 1) - \mathsf{E_b} \log \mathsf{E_b}.$$

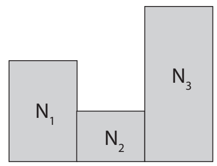
For $\mathsf{E_b} \ll 1$, the second term is the most significant term so that

$$\mathsf{C} \quad \approx \quad -\mathsf{E_b} \log_2 \mathsf{E_b}, \qquad \text{bits},$$

which is the small-signal limit of the entropy of a Poisson probability distribution (cf. (14.2.17)).

14.11 *Water filling and the capacity of multi-input multi-output channel*

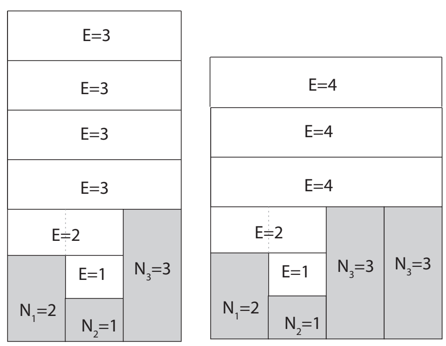
Consider a multi-input multi-output additive gaussian noise channel that supports three subchannels. The effective noise energy $N_k = N_0/\xi_k$ for each of these subchannels is shown in the figure below where ξ_k is the kth eigenvalue of the matrix $\mathbb{H}\mathbb{H}^\dagger$ with $\mathbb{H}$ being the channel matrix.



(a) Suppose that the total energy E available for transmission satisfies $E > \sum_{k=1}^{3} N_k$ where N_k is the effective noise power density spectrum in each subchannel. Graphically solve for the optimal energy allocation per subchannel using water filling and determine the optimal value of the energy E_k for each subchannel in terms of E and N_k.

Solution

The graphical solution is shown on the left side of the figure below with the total energy E available for transmission set to a value a convenient value of 15 in the same units as that for the noise energy.



(b) Determine the capacity C_k of each subchannel.

Solution

The capacity for each subchannel is given by (cf. (14.4.1))

$$\mathsf{C}_k = \log\left(1 + \frac{E_k}{N_k}\right)$$

Starting with the subchannel that has the most signal gives

$\mathsf{C}_1 = \log\left(1+\frac{6}{1}\right)$ = 2.8 bits
$\mathsf{C}_2 = \log\left(1+\frac{5}{2}\right)$ = 1.81 bits
$\mathsf{C}_3 = \log\left(1+\frac{4}{3}\right)$ = 1.22 bits

(c) Determine the overall capacity.

Solution
The total capacity C is just the sum of the capacities for each subchannel with $\mathsf{C} = \mathsf{C}_1 + \mathsf{C}_2 + \mathsf{C}_3 = 5.84$ bits.

(d) Now suppose that there is a fourth subchannel with an effective noise N_4. Determine the capacity for this system. Compare this result to the result for the three-subchannel system using the same total energy.

Solution
Water-filling the four subchannels, which is shown on the right side of the figure, the capacity for each subchannel is
$\mathsf{C}_1 = \log\left(1+\frac{5}{1}\right)$ = 2.58 bits
$\mathsf{C}_2 = \log\left(1+\frac{4}{2}\right)$ = 1.58 bits
$\mathsf{C}_3 = \log\left(1+1\right)$ = 1 bit
$\mathsf{C}_4 = \log\left(1+1\right)$ = 1 bit
with the total capacity equal to by $\mathsf{C} = \mathsf{C}_1 + \mathsf{C}_2 + \mathsf{C}_3 + \mathsf{C}_4 = 6.17$ bits. This result shows that even when an additional subchannel is noisy, it is better to allocate some energy to that subchannel rather than to ignore the noisy subchannel.

14.12 *Maximum information rate in terms of the arrival rate*

The expression for the maximum information rate of an ideal photon-optics channel is given by (cf. (14.4.27))

$$\mathcal{C} = f_{\text{max}} \frac{\pi^2}{\log_e 8},$$

and is expressed in terms of the frequency f_{max} of a photon with average energy E. In this problem, an equivalent expression is derived expressing the capacity in terms of the signal power P_s by relating the arrival rate of signal photons f_{max} to the signal power P_s.
(a) Integrate the power density spectrum

$$\mathcal{S}(f) = hf\left(\frac{1}{e^{hf/E}-1}\right),$$

over an infinite bandwidth to derive an expression that relates the total signal power P_s to the energy E. The definite integral $\int_0^\infty x/(e^x+1)\mathrm{d}x = \pi^2/6$ will be useful.

Solution
Let $x = hf/E$ so that $hf = Ex$ and $\mathrm{d}f = (E/h)\mathrm{d}x$. Making these substitutions into the expression for $\mathcal{S}(f)$, the total signal power P_s is

$$\begin{aligned} P_s &= \frac{E^2}{h}\int_0^{\infty} \frac{x}{e^x+1}\mathrm{d}x \\ &= \frac{\pi^2 E^2}{6h}. \end{aligned}$$

(b) Use the expression derived in part (a) to show that $f_{\max} = E/h$ is equal to $\sqrt{6P_s/(\pi^2 h)}$.

Solution
Rearrange the expression for P_s derived in part (a) to give

$$\frac{E^2}{h^2} = \frac{6P_s}{\pi^2 h}.$$

Using $E/h \doteq f_{\max}$ and taking the square-root of each side gives

$$\frac{E}{h} \doteq f_{\max} = \sqrt{\frac{6P_s}{\pi^2 h}}.$$

(c) Substitute the expression derived in part (b) into the expression for the bandlimited capacity to show that

$$\mathcal{C} = \frac{\pi}{\log_e 2}\sqrt{\frac{2P_s}{3h}} \qquad \text{bits per second,}$$

which is (14.4.30).

Solution
Using the expression for the bandlimited capacity and substituting $f_{\max} = \sqrt{6P_s/\pi^2 h}$ gives

$$\begin{aligned} \mathcal{C} &= \sqrt{\frac{6P_s}{\pi^2 h}}\,\frac{\pi^2}{\log_e 8} \\ &= \frac{\pi}{\log_e 2}\sqrt{\frac{2P_s}{3h}} \qquad \text{bits per second,} \end{aligned}$$

where $\log_e 8 = 3\log_e 2$ has been used. This expression is (14.4.30).

14.14 *Bandlimited capacity*

The bandlimited capacity for a wave-optics channel can be written in terms of a scaled wave-optics information rate $R_\mathrm{w} \doteq RE_b/N_0$

$$\mathcal{C} \;=\; B\log_2\left(1+R_\mathrm{w}/B\right).$$

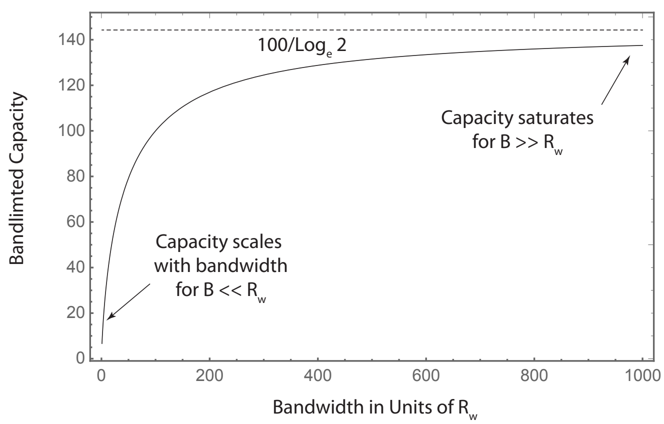
(a) Using a small-signal expansion of this expression, show that when B is much smaller than R_w, the bandlimited capacity scales linearly in B.

Solution

When B is much smaller than R_w, the term R_w/B is much greater than one. Therefore, the value of one inside the argument of the logarithm function can be ignored. This gives

$$\begin{aligned}\mathcal{C} &\approx B\log_2\left(R_\mathrm{w}/B\right)\\ &\approx B\log_2 R_\mathrm{w} - B\log_2 B.\end{aligned}$$

When B is much smaller than R_w, the first term dominates and the capacity is nearly linear in the bandwidth B for $B \ll R_\mathrm{w}$ as is shown in the figure below with $R_\mathrm{w} = 100$.



(b) Using a large-signal expansion of this expression, show that when B is much larger than R_w, the bandlimited capacity saturates to a value given by $R_\mathrm{w}/\log_e 2$.

Solution

Using the small argument expansion of a log function

$$\log(1+x) \approx x,$$

with $x = R_{\mathrm{w}}/B$ and a base two for the logarithm gives

$$B \log(1 + R_{\mathrm{w}}/B) \approx R_{\mathrm{w}}/\log_e 2.$$

This limiting value is also shown in the figure.

14.15 *Photon-optics spectral rate efficiency*

The spectral rate efficiency of a photon-optics channel including additive noise satisfies the following inequality (cf. (14.2.6) and (14.5.1))

$$r \leq \log_2\left(1 + \frac{r\mathsf{E}_\mathsf{b}}{1 + \mathsf{N}_0}\right) + (r\mathsf{E}_\mathsf{b} + \mathsf{N}_0)\log_2\left(1 + \frac{1}{r\mathsf{E}_\mathsf{b} + \mathsf{N}_0}\right) - \mathsf{N}_0 \log_2\left(1 + \frac{1}{\mathsf{N}_0}\right).$$

(a) Expand the right side of this expression in a power series in $r\mathsf{E}_\mathsf{b}$ up to the linear term.

Solution

Using $\log_e$ and expanding the right side in a power series in $r\mathsf{E}_\mathsf{b}$ gives

$$\left(\frac{1}{\mathsf{N}_0} - \frac{1}{1 + \mathsf{N}_0} + \log_e\left(1 + \frac{1}{\mathsf{N}_0}\right)\right) r\mathsf{E}_\mathsf{b}.$$

(b) Set the expression derived in part (a) equal to r and solve for E_b in terms of N_0. This expression can be used to determine $(\mathsf{E}_\mathsf{b})_{\min}$ when r equals zero.

Solution

Setting the expression derived in part (a) equal to r gives

$$\frac{1}{\mathsf{N}_0} - \frac{1}{1 + \mathsf{N}_0} + \log_e\left(1 + \frac{1}{\mathsf{N}_0}\right) = \frac{1}{(\mathsf{E}_\mathsf{b})_{\min}}.$$

This expression is the minimum value for $(\mathsf{E}_\mathsf{b})_{\min}$ given a value for N_0.

(c) Set $\mathsf{N}_0 = 1$ in the expression derived in part (b) and show that $(\mathsf{E}_\mathsf{b})_{\min}$ is equal to $\log_e 2/(1/2 + \log_e 2)$. This expression is a factor of $(1/2 + \log_e 2)$ smaller than $(E_b/N_0)_{\min} = \log_e 2$ for a wave-optics channel for the same rate expressed in bits.

Solution

Substituting $\mathsf{N}_0 = 1$ in the expression derived in part (b) and scaling the result by $\log_e 2$ to convert from nats to bits gives

$$(\mathsf{E}_\mathsf{b})_{\min} = \frac{\log_e 2}{1/2 + \log_e 2}.$$

This expression is a factor of $(1/2 + \log_e 2)$ smaller than $(E_b/N_0)_{\text{min}} = \log_e 2$ for a wave-optics channel.

(d) Show that when N_0 is much smaller than one, $(\mathsf{E}_\mathsf{b})_{\text{min}}$ goes as N_0.

Solution
Solving for $(\mathsf{E}_\mathsf{b})_{\text{min}}$ for the expression derived in part (b) gives

$$(\mathsf{E}_\mathsf{b})_{\text{min}} = \frac{\mathsf{N}_0(1+\mathsf{N}_0)}{1+\mathsf{N}_0\log(1+1/\mathsf{N}_0)+{\mathsf{N}_0}^2\log(1+1/\mathsf{N}_0)}.$$

The first term of the power series expansion of this expression is N_0. This means that for small N_0, the minimum mean number of photons $(\mathsf{E}_\mathsf{b})_{\text{min}}$ required for reliable communication is linear in N_0.

Chapter 15 Selected Solutions

15.2 *Coherent states as a basis*

Prove (15.3.37) by writing $\boldsymbol{\alpha}$ in polar coordinates and performing the resulting integrations using $\int_0^\infty re^{-r^2}r^{n+m}\mathrm{d}r = \frac{1}{2}\Gamma[(n+m+2)/2]$ where $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}\mathrm{d}x$ is the gamma function. Note that $\Gamma(j+1) = j!$ where j is an integer.[3]

Solution

The closure property given in (15.3.37) is

$$\widehat{I} = \frac{1}{\pi}\int_{\boldsymbol{\alpha}} |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\,\mathrm{d}\boldsymbol{\alpha}.$$

To proof this statement, write the Glauber number in polar coordinates as $\boldsymbol{\alpha} = \alpha e^{\mathrm{i}\phi}$ and $\mathrm{d}\boldsymbol{\alpha} = r\mathrm{d}r\mathrm{d}\phi$ where α is the magnitude of $\boldsymbol{\alpha}$ and ϕ is the phase. Using (15.3.34), the coherent state $|\boldsymbol{\alpha}\rangle$ in a photon-number state representation can be written as

$$|\boldsymbol{\alpha}\rangle = e^{-|\boldsymbol{\alpha}|^2/2}\sum_{\mathsf{m}=0}^{\infty}\frac{\boldsymbol{\alpha}^{\mathsf{m}}}{\sqrt{\mathsf{m}!}}|\mathsf{m}\rangle\,.$$

Similarly,

$$\langle\boldsymbol{\alpha}| = e^{-|\boldsymbol{\alpha}|^2/2}\sum_{\mathsf{n}=0}^{\infty}\frac{(\boldsymbol{\alpha}^*)^{\mathsf{n}}}{\sqrt{\mathsf{n}!}}\langle\mathsf{n}|.$$

Substituting these expressions into the integral gives

$$\begin{aligned}\frac{1}{\pi}\int_{\boldsymbol{\alpha}}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\,\mathrm{d}\boldsymbol{\alpha} &= \frac{1}{\pi}\sum_{\mathsf{m}=0}^{\infty}\sum_{\mathsf{n}=0}^{\infty}\frac{|\mathsf{m}\rangle\langle\mathsf{n}|}{\sqrt{\mathsf{m}!\mathsf{n}!}}\int_{\boldsymbol{\alpha}}e^{-|\boldsymbol{\alpha}|^2}\boldsymbol{\alpha}^{\mathsf{m}}(\boldsymbol{\alpha}^*)^{\mathsf{n}}\mathrm{d}\boldsymbol{\alpha}\\ &= \frac{1}{\pi}\sum_{\mathsf{m}=0}^{\infty}\sum_{\mathsf{n}=0}^{\infty}\frac{|\mathsf{m}\rangle\langle\mathsf{n}|}{\sqrt{\mathsf{m}!\mathsf{n}!}}\int_0^\infty e^{-r^2}r^{\mathsf{m}+\mathsf{n}}r\mathrm{d}r\int_0^{2\pi}e^{\mathrm{i}(\mathsf{m}+\mathsf{n})\phi}\mathrm{d}\phi.\end{aligned}$$

The integral on ϕ evaluates to $2\pi\delta_{\mathsf{mn}}$ where δ_{ij} is the Kronecker impulse. Setting n equal to m gives

$$\frac{1}{\pi}\int_{\boldsymbol{\alpha}}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\,\mathrm{d}\boldsymbol{\alpha} = \underbrace{\sum_{\mathsf{m}=0}^{\infty}|\mathsf{m}\rangle\langle\mathsf{m}|}_{\widehat{I}}\frac{2}{\mathsf{m}!}\int_0^\infty e^{-r^2}r^{2\mathsf{m}+1}\mathrm{d}r.$$

[3] Note that the problem stated $\Gamma(j-1) = j!$ where j is an integer. This is a typo. This should read $\Gamma(j+1) = j!$ where j is an integer.

Now use the formula given in the problem to write

$$\int_0^\infty e^{-r^2} r^{2\mathsf{m}+1} \mathrm{d}r \quad = \quad \tfrac{1}{2}\Gamma(\mathsf{m}+1) \quad = \quad \tfrac{1}{2}\mathsf{m}!.$$

Using this expression gives the closure relationship as

$$\frac{1}{\pi}\int_{\boldsymbol{\alpha}} |\boldsymbol{\alpha}\rangle \langle\boldsymbol{\alpha}| \, \mathrm{d}\boldsymbol{\alpha} \quad = \quad \widehat{I}.$$

15.5 *Eigenstates and eigenvalues of a quantized harmonic oscillator*

The differential equation

$$\frac{\mathrm{d}^2\phi_n(y)}{\mathrm{d}y^2} + \left(1 + 2n - y^2\right)\phi_n(y) \quad = \quad 0$$

where n is an integer has the solution

$$\phi_n(y) \quad = \quad H_n(y)e^{-y^2/2},$$

where $H_n(y)$ is the nth Hermite polynomial (cf. (3.3.50)). The functions are normalized so that

$$\int_{-\infty}^{\infty} \phi_n(y)\phi_m(y)\mathrm{d}y = 2^n\sqrt{\pi}n!\delta_{mn}$$

with δ_{mn} being the Kronecker impulse. Using this solution and the orthogonality relation, show that the eigenvalues of the quantized harmonic oscillator are given by

$$E \quad = \quad \hbar\omega\left(n + \frac{1}{2}\right)$$

which is (15.3.25) with the quantum wave function in the in-phase component representation given by

$$u_n(\alpha_I) \quad = \quad \sqrt{\frac{1}{2^n\sqrt{\pi}n!}}H_n(\alpha_I)e^{-\alpha_I^2/2}.$$

Solution

Compare the equation given in this problem to the Schrödinger equation for a quantized harmonic oscillator given in Problem 15.4, which is repeated here as

$$\frac{\mathrm{d}^2u(\alpha_I)}{\mathrm{d}a_I^2} + \left(2E/\hbar\omega - a_I^2\right)\psi(\alpha_I) \quad = \quad 0.$$

Examining the two equations we can associate y with a_I. Moreover, we can write

$$1 + 2n = \frac{2E}{\hbar\omega}$$

or

$$E = \hbar\omega\left(n + \frac{1}{2}\right).$$

These eigenvalues are the allowed energies. The eigenfunctions are

$$u_n(a_I) = K\, H_n(a_I)e^{-a_I^2/2},$$

where K is a normalization constant. Applying the normalization condition given in the problem gives

$$\int_{-\infty}^{\infty} u_n(a_I)u_n(a_I)\mathrm{d}a_I = 2^n\sqrt{\pi}n! = K^2,$$

or

$$K = \sqrt{2^n\sqrt{\pi}n!}$$

The normalized eigenfunctions are then given by

$$u(\alpha_I) = \sqrt{\frac{1}{2^n\sqrt{\pi}n!}}H_n(\alpha_I)e^{-\alpha_I^2/2}.$$

15.7 *Commutation relationships*

(a) Prove that $[\widehat{N}, \widehat{a}] = -\widehat{a}$ where $\widehat{N} = \widehat{a}^\dagger\widehat{a}$.

Solution

$$\begin{aligned}
[\widehat{N}, \widehat{a}] &= \widehat{a}^\dagger\widehat{a}\widehat{a} - \widehat{a}\widehat{a}^\dagger\widehat{a} \\
&= -\underbrace{\left(\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\right)}_{=1\ (15.3.8)}\widehat{a} \\
&= -\widehat{a}.
\end{aligned}$$

(b) Prove that $[\widehat{N}, \widehat{a}^\dagger] = \widehat{a}^\dagger$.

Solution

$$\begin{aligned}
[\widehat{N}, \widehat{a}^\dagger] &= \widehat{a}^\dagger\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\widehat{a}^\dagger \\
&= \widehat{a}^\dagger\underbrace{\left(\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a}\right)}_{=1\ (15.3.8)} \\
&= \widehat{a}^\dagger.
\end{aligned}$$

15.9 *Commutation in an enlarged signal space*

Let $\mathbb{A}$ and $\mathbb{B}$ be two n by n hermitian matrices for which $[\mathbb{A}, \mathbb{B}] \neq 0$.

(a) Prove that the two $2n$ by $2n$ matrices

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbb{B} & \mathbb{A} \\ \mathbb{A} & \mathbb{B} \end{bmatrix}$$

do commute. This shows that operators that do not commute can be embedded into a larger signal space, called an ancilla embedding, in which they do commute. This motivates the definition of an ancilla state.

Solution

Let

$$\mathbb{C} \quad = \quad \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix} \quad \text{and} \quad \mathbb{D} \quad = \quad \begin{bmatrix} \mathbb{B} & \mathbb{A} \\ \mathbb{A} & \mathbb{B} \end{bmatrix},$$

then

$$\begin{aligned}
[\mathbb{C}\mathbb{D}] \quad = \mathbb{C}\mathbb{D} - \mathbb{D}\mathbb{C} \quad &= \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix} \begin{bmatrix} \mathbb{B} & \mathbb{A} \\ \mathbb{A} & \mathbb{B} \end{bmatrix} - \begin{bmatrix} \mathbb{B} & \mathbb{A} \\ \mathbb{A} & \mathbb{B} \end{bmatrix} \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A} & \mathbb{A}^2 + \mathbb{B}^2 \\ \mathbb{B}^2 + \mathbb{A}^2 & \mathbb{B}\mathbb{A} + \mathbb{A}\mathbb{B} \end{bmatrix} - \begin{bmatrix} \mathbb{B}\mathbb{A} + \mathbb{A}\mathbb{B} & \mathbb{B}^2 + \mathbb{A}^2 \\ \mathbb{A}^2 + \mathbb{B}^2 & \mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A} \end{bmatrix} \\
&= 0
\end{aligned}$$

showing that a noncommuting tranformation in a smaller signal space can always be embedded in a larger signal space using ancilla states with the embedded transformation commuting in the larger signal space. For this reason, a generalized measurement defined using a set of noncommuting measurement operators $\{\widehat{Y}_n\}$ in a smaller signal space, which ignores how ancilla states may interact with the signal state during the measurement process, can always be expressed using a set of commuting measurement operators defined in a larger signal space defined using the ancilla states even when those states have no signal.

(b) Prove that $\text{trace}\big(\mathbb{A}\mathbb{B}\big) = \text{trace}\big(\mathbb{B}\mathbb{A}\big)$ (cf. 2.1.84c) even if $[\mathbb{A}, \mathbb{B}]$ is not equal to zero.

Solution
From the definition of the trace operation we have

$$\begin{aligned}
\text{trace}\big(\mathbb{A}\mathbb{B}\big) &= (\mathbb{A}\mathbb{B})_{11} + (\mathbb{A}\mathbb{B})_{22} + \cdots + (\mathbb{A}\mathbb{B})_{nn} \\
&= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \\
&+ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} \\
&\quad\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\
&+ a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}
\end{aligned}$$

and

$$\begin{aligned}
\text{trace}\big(\mathbb{B}\mathbb{A}\big) &= (\mathbb{B}\mathbb{A})_{11} + (\mathbb{B}\mathbb{A})_{22} + \cdots + (\mathbb{B}\mathbb{A})_{nn} \\
&= b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} \\
&+ b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2} \\
&\quad\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\
&+ b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}
\end{aligned}$$

Each of these summations contains the same set of terms, which can be seen by transposing the rows and columns of one of the summations. Therefore, $\text{trace}\big(\mathbb{A}\mathbb{B}\big) = \text{trace}\big(\mathbb{B}\mathbb{A}\big)$.

15.10 *Phase operator*

Consider a phase operator $\widehat{e^{\mathrm{i}\phi}}$ defined by the two equations

$$\begin{aligned}
\widehat{a} &= \sqrt{\widehat{N}+1}\,\widehat{e^{\mathrm{i}\phi}} \\
\widehat{a}^\dagger &= \widehat{e^{-\mathrm{i}\phi}}\sqrt{\widehat{N}+1}.
\end{aligned}$$

(a) Write the two phase operators $\widehat{e^{\mathrm{i}\phi}}$ and $\widehat{e^{-\mathrm{i}\phi}}$ in terms of $\widehat{a}$, $\widehat{a}^\dagger$, and $\widehat{N}$.

Solution
Using $\widehat{N} = \widehat{a}^\dagger\widehat{a}$ and $[\widehat{a}, \widehat{a}^\dagger] = 1$, the term $\widehat{N}+1$ is equal to $\widehat{a}\widehat{a}^\dagger$. Using this expression, the two phase operators can be formally written as[4]

$$\begin{aligned}
\widehat{e^{\mathrm{i}\phi}} &= (\widehat{N}+1)^{-1/2}\widehat{a} &= (\widehat{a}\widehat{a}^\dagger)^{-1/2}\widehat{a} \\
\widehat{e^{-\mathrm{i}\phi}} &= \widehat{a}^\dagger(\widehat{N}+1)^{-1/2} &= \widehat{a}^\dagger(\widehat{a}\widehat{a}^\dagger)^{-1/2}.
\end{aligned}$$

[4]For further details see L. Susskind and J. Glogower,"Quantum mechanical phase and time operator". Physica, 1: 49, 1964.

(b) Prove that $[\widehat{N}, \widehat{e^{\mathrm{i}\phi}}] = -\widehat{e^{\mathrm{i}\phi}}$. (Note that the problem statement was missing a negative sign on the second expression.)

Solution

Compare the expressions for the phase operators to the lowering operator given in (15.3.29) and repeated below

$$\widehat{a}\,|\mathsf{m}\rangle \quad = \quad \mathsf{m}^{-1/2}\,|\mathsf{m}-1\rangle$$

and the and raising operator given in (15.3.30) and repeated below

$$\widehat{a}^{\dagger}\,|\mathsf{m}\rangle \quad = \quad (\mathsf{m}+1)^{-1/2}\,|\mathsf{m}+1\rangle\,.$$

The normalization constants given in these equations can be incorporated by treating $\widehat{e^{-\mathrm{i}\phi}}$ as a normalized raising operator and $\widehat{e^{\mathrm{i}\phi}}$ as a normalized lower operator with

$$\begin{aligned}\widehat{e^{\mathrm{i}\phi}}\,|\mathsf{m}\rangle &= |\mathsf{m}-1\rangle\\ \widehat{e^{-\mathrm{i}\phi}}\,|\mathsf{m}\rangle &= |\mathsf{m}+1\rangle\,.\end{aligned}$$

Now use the correspondance between $\widehat{e^{\mathrm{i}\phi}}$ and $\widehat{a}$ and $\widehat{e^{-\mathrm{i}\phi}}$ and $\widehat{a}^{\dagger}$, and the results of Problem 7 to write

$$[\widehat{N}, \widehat{a}^{\dagger}] \quad = \quad \widehat{a}^{\dagger}\widehat{a}\widehat{a}^{\dagger} - \widehat{a}^{\dagger}\widehat{a}\widehat{a}^{\dagger} \quad = \quad \widehat{a}^{\dagger}\left(\widehat{a}\widehat{a}^{\dagger} - \widehat{a}^{\dagger}\widehat{a}\right) \quad = \quad \widehat{a}^{\dagger}$$

and

$$[\widehat{N}, \widehat{a}] \quad = \quad \widehat{a}^{\dagger}\widehat{a}\widehat{a} - \widehat{a}\widehat{a}^{\dagger}\widehat{a} \quad = \quad -\left(\widehat{a}\widehat{a}^{\dagger} - \widehat{a}^{\dagger}\widehat{a}\right)\widehat{a} \quad = \quad -\widehat{a}.$$

Using these expressions leads to

$$[\widehat{N}, \widehat{e^{-\mathrm{i}\phi}}] \quad = \quad \widehat{e^{-\mathrm{i}\phi}}$$

and

$$[\widehat{N}, \widehat{e^{\mathrm{i}\phi}}] \quad = \quad -\widehat{e^{\mathrm{i}\phi}}.$$

15.16 *Shannon and von Neumann entropies*

A two-by-two density matrix is given by

$$\widehat{\rho} \quad = \quad \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}.$$

Compute the Shannon entropy and the von Neumann entropy. Comment.

Solution
When appropriate, the Shannon entropy is simply the entropy of the diagonal elements of the matrix and is given by

$$\begin{aligned} H &= \sum_n p_n \log p_n = 0.5 \log 0.5 + 0.5 \log 0.5 = \log_2 0.5 \\ &= 1 \text{ bit} \end{aligned}$$

because the equiprobable prior $p = 1/2$ achieves the capacity of a binary symmetric channel.

The von Neumann entropy is the Shannon entropy of the eigenvalues of the density matrix $\widehat{\rho}$, which are $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$. Therefore

$$\begin{aligned} H &= \sum_n \lambda_n \log \lambda_n = 0.6 \log 0.6 + 0.4 \log 0.4 = 0.6 \log_2 0.6 + 0.4 \log_2 0.4 \\ &= 0.971 \text{ bits} \end{aligned}$$

showing that the von Neumann entropy is less than the Shannon entropy whenever the matrix is a nondiagonal matrix. Physically, the nondiagonal nature of the density matrix indicates that the system has quantum uncertainty in the chosen measurement basis used to express the density matrices.

15.18 *Partial trace*

The partial trace of the product state $\widehat{\rho} = \widehat{\sigma} \otimes \widehat{\mu}$ that recovers the density matrix of the component signal state $\widehat{\mu}$ is given by (15.4.9a). Determine an explicit expression for the partial trace of $\widehat{\rho}$ that recovers the density matrix of the component signal state $\widehat{\sigma}$.

Solution
Let $\widehat{\sigma} = \widehat{\rho}_A$ and let $\widehat{\mu} = \widehat{\rho}_B$. These density matrices are formed using two signal states $|A\rangle$ and $|B\rangle$. Suppose that one state is an element of a signal space $\mathcal{A}$ spanned by one set of basis states $\{|a_i\rangle\}$ so that

$$|A\rangle = \sum a_i |a_i\rangle.$$

Similarly, $|B\rangle$ is an element of a signal space $\mathcal{B}$ spanned by different set of basis states $\{|b_i\rangle\}$

$$|B\rangle = \sum b_k |b_k\rangle.$$

Let the density matrix $\widehat{\rho}_{AB} = \widehat{\rho}_A \otimes \widehat{\rho}_B$ of a composite signal state be the outer product of the two constituent signal states given by

$$\widehat{\rho}_{AB} \quad = \quad \sum_{ijk\ell} a_{ij} b_{k\ell} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_\ell|.$$

The partial trace over the basis states of the constituent signal space $\mathcal{B}$ is given by

$$\begin{aligned} \text{trace}_B \widehat{\rho}_{AB} &= \sum_{ijk\ell} a_{ij} b_{k\ell} |a_i\rangle\langle a_j| \otimes \text{trace}|b_k\rangle\langle b_\ell| \\ &= \sum_{ijk\ell} a_{ij} b_{k\ell} |a_i\rangle\langle a_j| \langle b_\ell | b_k\rangle \end{aligned}$$

where the trace converts the outer product to an inner product (cf. (2.1.85)). Similarly,

$$\text{trace}_A \widehat{\rho}_{AB} \quad = \quad \sum_{ijk\ell} a_{ij} b_{k\ell} |b_k\rangle\langle b_\ell| \otimes \text{trace}|a_i\rangle\langle a_j| \quad = \quad \sum_{ijk\ell} a_{ij} b_{k\ell} |b_k\rangle\langle b_\ell| \langle a_i | a_j\rangle.$$

When expressed using matrices, $\widehat{\rho}_{AB}$ is given by the Kronecker product (cf. (2.1.99))

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad = \quad \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}, \quad (38)$$

which can be written as

$$\begin{bmatrix} a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix}.$$

The partial trace $\text{trace}_A \widehat{\rho}_{AB}$ over the constituent signal space $\mathcal{A}$ is then

$$\begin{aligned} \text{trace}_A \widehat{\rho}_{AB} &= a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= (a_{11} + a_{22}) \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \widehat{\rho}_B, \end{aligned}$$

where $\text{trace}\widehat{\rho}_A = 1$ has been used. Similarly, $\text{trace}_B\widehat{\rho}_{AB}$ over the constituent signal space $\mathcal{B}$ is

$$
\begin{aligned}
\text{trace}_B\widehat{\rho}_{AB} &= \begin{bmatrix} a_{11}\,\text{trace}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\,\text{trace}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21}\,\text{trace}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\,\text{trace}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&= \widehat{\rho}_A,
\end{aligned}
$$

where $\text{trace}\widehat{\rho}_B = 1$ has been used. These expressions show that the partial trace for a product state marginalizes the state because it "traces out" the density matrices of the other constituent signals states in the product state. In this sense, the partial trace operation for product states is equivalent to the marginalization of a classical product distribution. This statement only applies to product states.

As a side note, the partial trace can be generalized to nonproduct states in the composite signal space by writing

$$
\text{trace}_B\widehat{\rho}_{AB} = \sum_{ijk\ell} c_{ijk\ell}|a_i\rangle\langle a_j|\text{trace}|b_k\rangle\langle b_\ell|
$$

where $c_{ijk\ell}$ need not factor into a product of the form $a_{ij}b_{k\ell}$ as would be the case for a product state.

Now consider the partial trace of an arbitrary density matrix $\widehat{\rho}$ in a composite signal space given by

$$
\widehat{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix}.
$$

Noting the elements in (38) that are used to construct the partial trace over each constituent signal space gives

$$
\begin{aligned}
\text{trace}_B\widehat{\rho} &= \begin{bmatrix} \text{trace}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} & \text{trace}\begin{bmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{bmatrix} \\ \text{trace}\begin{bmatrix} \rho_{31} & \rho_{32} \\ \rho_{41} & \rho_{42} \end{bmatrix} & \text{trace}\begin{bmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{bmatrix}.
\end{aligned}
$$

Similarly, noting that a different set of matrix elements is used for $\text{trace}_B\widehat{\rho}$ gives

$$\begin{aligned}\text{trace}_A\widehat{\rho} &= \begin{bmatrix} \text{trace}\begin{bmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{bmatrix} & \text{trace}\begin{bmatrix} \rho_{12} & \rho_{14} \\ \rho_{32} & \rho_{34} \end{bmatrix} \\ \text{trace}\begin{bmatrix} \rho_{21} & \rho_{23} \\ \rho_{41} & \rho_{43} \end{bmatrix} & \text{trace}\begin{bmatrix} \rho_{22} & \rho_{24} \\ \rho_{42} & \rho_{44} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{bmatrix}.\end{aligned}$$

These expressions show that for the general case, a partial trace does not recover one of the constituent signal states and is not equivalent to marginalization. This situation can occur when a unitary transformation is applied to a product signal state. Constructing a composite signal state from constituent signal states, applying unitary transformations and then applying a partial trace on a constituent signal space are the basic operations of quantum computing.

15.19 *Large-signal regime*

Suppose that a coherent state is regarded as orthogonal when the pairwise inner product $\kappa = e^{-4\mathsf{E}_b}$ is 10^{-5} where E_b is the mean number of photons.

(a) Determine the signal power density spectrum required to achieve this condition for the following wavelengths: 1 nm, 1000 nm (1 micron), 10^6 nm (1 mm).

Solution
Solving $e^{-4\mathsf{E}_b} = 10^{-5}$ gives E_b as 2.88 photons. The energy per photon is equal to $hf = hc/\lambda$. The signal power density spectrum is equal to the energy which is given by energy/photon$\times$mean number of photons. The energy per photon is equal to $hf = hc/\lambda$. Using this expression gives

$$\begin{aligned}\mathcal{S}(1\text{ nm}) &= \mathsf{E}_b hc/10^{-9} &= 1.9878\times10^{-16}\text{ W/Hz}\\ \mathcal{S}(1\text{ micron}) &= \mathsf{E}_b hc/10^{-6} &= 1.9878\times10^{-19}\text{ W/Hz}\\ \mathcal{S}(1\text{ mm}) &= \mathsf{E}_b hc/10^{-3} &= 1.9878\times10^{-22}\text{ W/Hz}\end{aligned}$$

(b) Determine thermal noise power density spectrum at 290 K given by $N_0 = kT_0$ where k is Boltzmann's constant and T_0 is the temperature in Kelvin.

Solution
The thermal energy is $N_0 = kT_0 = 290\times1.38\times10^{-23} = 4\times10^{-21}$ J (or W/Hz), which is larger than the signal power density spectrum at 1 mm, but is much smaller compared to the signal power density spectrum for lightwave or higher frequencies.

(c) Suppose that the nonorthogonal nature of the coherent states is evident when κ is larger than one half and that the signal to thermal noise ratio E/N_0 is larger than 20 dB. For these parameters, what is the largest wavelength for which the nonorthogonal nature of a coherent state is evident?

Solution
Solving $e^{-4\mathsf{E}_\mathsf{b}} = 1/2$ gives the mean number of photons as $\mathsf{E}_\mathsf{b} = 0.17$. The signal energy when E/N_0 is equal to 20 dB with $N_0 = kT_0$ determined in part (b) is

$$\mathcal{S} = 100N_0 = 4 \times 10^{-19} \text{ J.}$$

The largest wavelength λ_{max} for which the nonorthogonal nature of a coherent state may be evident is then

$$\lambda_{\text{max}} = \frac{hc\mathsf{E}_\mathsf{b}}{\mathcal{S}} \approx 86 \text{ nm.}$$

This expression states for the value of κ and E/N_0 given in the problem, the nonorthogonality of the coherent states would not be evident for optical wavelengths on the order of 0.4 to 1.5 microns because for this range of wavelength, the mean number of photons E is sufficiently large so the nonorthogonality of the signal states is not evident. For the same parameters, the nonorthogonality might be evident in ultraviolet regime of the spectrum within the range of 10-400 nm because for the same mean energy, there are, on average, fewer photons.

15. 20 *Noise from phase-insensitive amplification*

Let the two operators

$$\begin{aligned} \widehat{x} &= \widehat{a}_I + \widehat{n}_I \\ \widehat{y} &= \widehat{a}_Q + \widehat{n}_Q \end{aligned}$$

be noisy versions of the in-phase operator and the quadrature operator where the terms $\widehat{n}_I$ and $\widehat{n}_Q$ account for the additional noise from phase-insensitive amplification.

(a) Write down the necessary condition for $\widehat{x}$ and $\widehat{y}$ to be jointly observed without additional uncertainty.

Solution
For the variables described by the operators $\widehat{x}$ and $\widehat{y}$ to be measured without additional error

requires that $\widehat{x}$ and $\widehat{y}$ commute so that

$$\begin{aligned}
[\widehat{x}\widehat{y}] &= \widehat{x}\widehat{y} - \widehat{y}\widehat{x} \\
&= (\widehat{a}_I + \widehat{n}_I)(\widehat{a}_Q + \widehat{n}_Q) - (\widehat{a}_Q + \widehat{n}_Q)(\widehat{a}_I + \widehat{n}_I) \\
&= (\widehat{a}_I\widehat{a}_Q + \widehat{a}_I\widehat{n}_Q + \widehat{n}_I\widehat{a}_Q + \widehat{n}_I\widehat{n}_Q) - (\widehat{a}_Q\widehat{a}_I + \widehat{a}_Q\widehat{n}_I + \widehat{n}_Q\widehat{a}_I + \widehat{n}_Q\widehat{n}_I) \\
&= [\widehat{a}_I, \widehat{a}_Q] + [\widehat{a}_I, \widehat{n}_Q] + [\widehat{n}_I, \widehat{a}_Q] + [\widehat{n}_I, \widehat{n}_Q]\,. \qquad (39)
\end{aligned}$$

(b) Using this condition, solve for the relationship between $\widehat{n}_I$ and $\widehat{n}_Q$ such that the condition in part (a) is satisfied.

Solution

The two middle terms in (39) are zero because the signal and noise are independent so the operators describing these terms commute. However, in contrast to heterodyne demodulation (cf. (15.5.12) and (15.5.13)), the two remaining terms in (39) must be zero so that the operators commute. This means that two commutators $[\widehat{a}_I, \widehat{a}_Q]$ and $[\widehat{n}_I, \widehat{n}_Q]$ must go to zero. This condition occurs in a large signal regime for which the inherent dependancies between the signal and noise components are not evident. Therefore, the noisy in-phase quadrature lightwave components after phase insensitive amplification can be jointly observed without additional error.

15.24 *Even and odd coherent states*

(a) An even coherent state is defined as

$$|\mathsf{even}\rangle \doteq \frac{1}{\sqrt{N_+}}\left(|\boldsymbol{\alpha}\rangle + |{-\boldsymbol{\alpha}}\rangle\right),$$

where N_+ is a normalization constant. When $\langle\mathsf{even}|\mathsf{even}\rangle$ equals one, using (15.3.38), repeated here as

$$\langle\boldsymbol{\alpha}_1|\boldsymbol{\alpha}_0\rangle = e^{-|\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_0|^2/2},$$

show that $N_+ = 2(1 + e^{-2|\alpha|^2})$.

Solution

Using the normalization condition gives

$$\begin{aligned}
\langle\mathsf{even}|\mathsf{even}\rangle &= \frac{1}{N_+}(((\langle\boldsymbol{\alpha}| + \langle-\boldsymbol{\alpha}|)((|\boldsymbol{\alpha}\rangle + |{-\boldsymbol{\alpha}}\rangle)) \\
&= \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle + \langle\boldsymbol{\alpha}|{-\boldsymbol{\alpha}}\rangle + \langle-\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle + \langle-\boldsymbol{\alpha}|{-\boldsymbol{\alpha}}\rangle
\end{aligned}$$

The first and last terms evaluate to one. Using the expression for the inner product stated in the problem with $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_0 = -\boldsymbol{\alpha}$ gives

$$\begin{aligned}\langle \boldsymbol{\alpha} | -\boldsymbol{\alpha} \rangle &= e^{-|\boldsymbol{\alpha}-(-\boldsymbol{\alpha})|^2/2} \\ &= e^{-|2\boldsymbol{\alpha}|^2/2}.\end{aligned}$$

The same expression is obtained for $\langle -\boldsymbol{\alpha} | \boldsymbol{\alpha} \rangle$. Combining the these expressions gives

$$N_+ = 2\left(1 + e^{-|2\boldsymbol{\alpha}|^2/2}\right).$$

(b) Repeat for the odd coherent state given by

$$|\mathsf{odd}\rangle = \frac{1}{\sqrt{N_-}}\left(|\boldsymbol{\alpha}\rangle - |-\boldsymbol{\alpha}\rangle\right).$$

Solution

The only difference is a negative sign for the two "cross terms" which gives

$$N_- = 2\left(1 - e^{-|2\boldsymbol{\alpha}|^2/2}\right).$$

(c) Derive approximate expressions for the even and odd coherent states when α is large and comment on the result.

Solution

The addition and subtraction operations on the coherent states that specify the even and odd coherent states are defined in the signal space for which coherent states are defined. These operations are not defined on the complex plane where the Glauber numbers are defined. For this reason, it is not correct to represent each large-amplitude coherent state as a single complex number $\mathbf{s}$ and then add the complex numbers. Doing so would result in the even coherent state being identically zero. To show this explicitly, use (15.3.34) to express both $|\boldsymbol{\alpha}\rangle$ and $|-\boldsymbol{\alpha}\rangle$ in a photon number state representation. The sum can be written as

$$\begin{aligned}|\mathsf{even}\rangle &= \frac{1}{\sqrt{N_+}}\left(|\boldsymbol{\alpha}\rangle + |-\boldsymbol{\alpha}\rangle\right) \\ &= \frac{1}{\sqrt{N_+}}\left(e^{-|\boldsymbol{\alpha}|^2/2}\sum_{\mathsf{m}=0}^{\infty}\frac{\boldsymbol{\alpha}^{\mathsf{m}}}{\sqrt{\mathsf{m}!}}|\mathsf{m}\rangle + e^{-|-\boldsymbol{\alpha}|^2/2}\sum_{\mathsf{m}=0}^{\infty}\frac{(-\boldsymbol{\alpha})^{\mathsf{m}}}{\sqrt{\mathsf{m}!}}|\mathsf{m}\rangle\right) \\ &= \frac{e^{-|\boldsymbol{\alpha}|^2}}{\sqrt{N_+}}\sum_{\mathsf{m}=0}^{\infty}\frac{\boldsymbol{\alpha}^{\mathsf{m}} + (-\boldsymbol{\alpha})^{\mathsf{m}}}{\sqrt{\mathsf{m}!}}|\mathsf{m}\rangle.\end{aligned}$$

For m even $\boldsymbol{\alpha}^{\mathsf{m}} = (-\boldsymbol{\alpha})^{\mathsf{m}}$ and for m odd $\boldsymbol{\alpha}^{\mathsf{m}} = -(-\boldsymbol{\alpha})^{\mathsf{m}}$. Therefore,

$$\begin{aligned} |\mathsf{even}\rangle &= \frac{2e^{-|\boldsymbol{\alpha}|^2}}{\sqrt{N_+}} \sum_{\mathsf{m}=0}^{\infty} \frac{\boldsymbol{\alpha}^{\mathsf{m}}}{\sqrt{\mathsf{m}!}} |\mathsf{m}\rangle \qquad \text{for } \mathsf{m} \text{ even} \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{for } \mathsf{m} \text{ odd,} \end{aligned}$$

which does not go to zero for large values of $|\boldsymbol{\alpha}|$. A similar expression can be derived for the odd coherent states.

Chapter 16 Selected Solutions

16.2 *Probability of error for orthogonal states*

This problem compares classical orthogonal signals to quantum orthogonal signal states. An example of orthogonal quantum-lightwave signals are polarization states or nonoverlapping temporal states. Derive the large-signal limit for the probability of a detection error for L-level orthogonal state modulation when all pairwise distances between the symbols are equal and $d^2 = 2\mathsf{E}$ where E is the mean number of photons per symbol.

Solution

The probability of a detection error in a large-signal limit can be estimated using the quantum union bound given in (16.3.8) and repeated here

$$p_e \approx \tfrac{1}{4}\overline{n}e^{-d^2_{\text{min}}},$$

where $\overline{n} = \frac{1}{L}\sum_{\ell=0}^{L-1} n_\ell$ is the average number of coherent-state symbols at distance d_{min}, where the distance is defined using the corresponding Glauber numbers for the coherent states (cf. (10.2.13)). When the signaling states are not coherent states but are still nearly orthogonal, the term $e^{-d^2_{\text{min}}}$ is replaced by the minimum pairwise inner product $\kappa_{\text{min}} \doteq \min_{i,j}\kappa_{ij}$. For L-level orthogonal state modulation, there are $L-1$ nearest neighbor symbol states. For an orthogonal signal-state constellation, each of these signal states has the same minimum pairwise inner product κ_{min} with every other signal state. Therefore,

$$\begin{aligned} p_e &\approx \frac{L-1}{4}\kappa_{\text{min}} \\ &\approx \frac{L-1}{4}e^{-2\mathsf{E}}, \end{aligned}$$

where $\kappa_{\text{min}} = e^{-d^2_{\text{min}}}$ for coherent-state symbols with $d^2_{\text{min}} = 2\mathsf{E}$ and E given as the mean number of photons per symbol.

16.3 *Optimal orientation of a binary sampling basis*

(a) Referring to Figure 16.2(a), define $\zeta = \pi/2 - \theta$. Derive an expression for the probability of a correct decision p_c for binary pure-signal-state modulation in terms of ζ and the generalized angle ϕ_1 shown in Figure 16.2(a).

Solution

Referring Figure 16.2(a), $\zeta = \pi/2 - \theta = \phi_0 + \phi_1$ so that $\phi_0 = \zeta + \phi_1$. Using the conditional probabilities $p_{0|0} = \cos^2\phi_0$ and $p_{1|1} = \cos^2\phi_1$, the probability for a correct detection event p_c is

$$\begin{aligned} p_c &= p\cos^2\phi_1 + (1-p)\cos^2\phi_0 \\ &= p\cos^2\phi_1 + (1-p)\cos^2(\pi/2-\theta-\phi_1). \end{aligned}$$

(b) Determine the maximum probability of a correct decision p_c by differentiating this expression with respect to ϕ_1 and setting the resulting expression equal to zero.

Solution

For an equiprobable prior $p = 1/2$ and the derivative of p_c with respect to ϕ_1 is given by

$$\frac{\mathrm{d}p_c}{\mathrm{d}\phi_1} = \sin(\theta+\phi_1)\cos(\theta+\phi_1) - \sin\phi_1\cos\phi_1.$$

Setting this expression equal to zero gives the optimal value of ϕ_1 as

$$\phi_1 = \pi/4 - \theta/2.$$

When $\theta = \pi/2$, the two signal states are orthogonal and $\phi_1 = \phi_0 = 0$. When $\theta = 0$, the two signal states are coincident $\phi_1 = \phi_0 = \pi/4$ and $p_c = p_e = 1/2$. For any value of θ, p_c is maximized and p_e is minimized when $\phi_0 = \phi_1$ so that the channel is a binary symmetric channel with $p_{1|1} = p_{0|0}$.

(c) Show that the resulting probability of error p_e is given by the same expression as (16.2.12).

Solution

For an equiprobable prior with $\phi_1 = \phi_0$, the channel is a binary symmetric channel with $p_{1|1} = p_{0|0}$. Then

$$\begin{aligned} p_c &= \tfrac{1}{2}\left(p_{1|1} + p_{0|0}\right) \quad = \quad p_{1|1} \\ &= \cos^2\phi_1 \\ &= \cos^2(\pi/4-\theta/2) \\ &= \tfrac{1}{2}\left(1+\cos(\pi/2-\theta)\right) \\ &= \tfrac{1}{2} + \tfrac{1}{2}\sin\theta. \end{aligned}$$

The probability of error is then

$$\begin{aligned} p_e &= 1 - p_c \\ &= \tfrac{1}{2}\left(1-\sin\theta\right), \end{aligned}$$

which is (16.2.12).

16.6 *Methods of detection*

Referring to Table 16.1, compare the error performance of classical homodyne detection with that of a displacement receiver. Determine the range of the mean number of photons per bit $\mathsf{E_b}$ for which one detection technique outperforms the other method of detection.

Solution

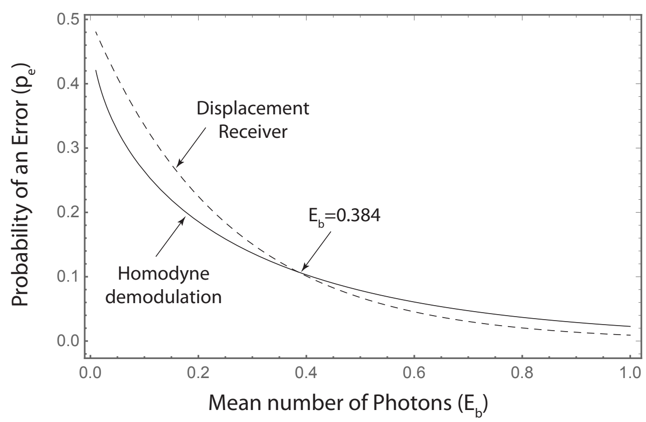
Referring to Table 16.1, the probability of a detection error for classical shot-noise-limited homodyne demodulation is

$$p_e \quad = \quad \tfrac{1}{2}\text{erfc}\sqrt{2\mathsf{E_b}},$$

and that for the displacement receiver is given by

$$p_e \quad = \quad \tfrac{1}{2}e^{-4\mathsf{E_b}}.$$

Plots of both functions are shown in the figure below. For $\mathsf{E_b} < 0.384$, shot-noise-limited homodyne demodulation produces a lower probability of a detection error than displacement demodulation. For $\mathsf{E_b} > 0.384$, the displacement receiver has better performance.



16.7 *Von Neumann entropy*

Suppose the density matrix of an ensemble of two pure signal states is given as

$$\widehat{\rho} \quad \doteq \quad \begin{bmatrix} p & \sqrt{p(1-p)}\kappa \\ \sqrt{p(1-p)}\kappa & 1-p \end{bmatrix},$$

where p is the prior, and where $\kappa \doteq \langle\psi_0|\psi_1\rangle$ is the inner product between the two pure states with κ real.

(a) Determine an expression for the von Neumann entropy of this density matrix.

Solution
The eigenvalues of the density matrix $\widehat{\rho}$ are the solutions to

$$\det \begin{bmatrix} p-\lambda & \sqrt{p(1-p)}\kappa \\ \sqrt{p(1-p)}\kappa & 1-p-\lambda \end{bmatrix} = 0,$$

or

$$\lambda^2 - \lambda + \kappa^2 p^2 - p^2 - \kappa^2 p + p = 0.$$

The solution to this equation gives the eigenvalues as

$$\lambda_{0,1} = \tfrac{1}{2}\Big(1 \pm \sqrt{1+4p\big(\kappa^2(1-p)+p-1\big)}\Big).$$

(b) Compare this result with the von Neumann entropy of the density matrix given in (15.4.16). How are they related? Why?

Solution
The eigenvalues are the same because the two density matrices are related by a unitary transformation. This means that the two density matrices are expressed in two different bases related by a generalized rotation described by the unitary transformation. This rotation does not affect the eigenvalues or the von Neumann entropy.

16.8 *Von Neumann entropy*

Using the relationship between a density matrix and a probability distribution for a set of orthogonal signal states, show that when the signal states are pairwise orthogonal, the Holevo information χ given in (16.5.9) and repeated below

$$\chi \doteq \mathcal{S}\Big(\sum_{\mathsf{s}} p(\mathsf{s})\widehat{\rho}_{\mathsf{s}}\Big) - \sum_{\mathsf{s}} p(\mathsf{s})\mathcal{S}\left(\widehat{\rho}_{\mathsf{s}}\right),$$

is equal to the Shannon entropy $H(\underline{\mathsf{s}})$.

Solution
For a constellation of orthogonal pure signal states, the density matrix $\widehat{\rho}_{\mathsf{s}}$ for each pure signal state has no quantum uncertainty, and has a single eigenvalue equal to one. Therefore

$\mathcal{S}(\widehat{\rho}_{\mathsf{s}}) = 0$. This density matrix $\widehat{\rho}_{\mathsf{s}}$ is a projection matrix $\widehat{P}_{\mathsf{s}} = |\psi_{\mathsf{s}}\rangle\langle\psi_{\mathsf{s}}|$ which can be written as

$$\widehat{P}_{\mathsf{s}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

This matrix has one diagonal element equal to one and all other elements, both diagonal and off-diagonal, equal to zero. The summation $\widehat{\rho} = \sum_{\mathsf{s}} p(\mathsf{s})\widehat{\rho}_{\mathsf{s}}$ over all the pure states in the signal constellation can then be written as

$$\begin{aligned} \sum_{\mathsf{s}} p(\mathsf{s})\widehat{\rho}_{\mathsf{s}} &= p(1) \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} + p(2) \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} + \\ & \cdots \quad + p(\mathsf{n}) \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \\ &= \begin{bmatrix} p(1) & 0 & 0 & \dots & \dots & 0 \\ 0 & p(2) & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & p(\mathsf{n}) \end{bmatrix}, \end{aligned}$$

which is a diagonal matrix $\mathbb{D}$ with diagonal elements that comprise the probability distribution $p(\mathsf{s})$. For this diagonal matrix, the eigenvalues λ_i of the density matrix $\widehat{\rho} = \sum_{\mathsf{s}} p(\mathsf{s})\widehat{\rho}_{\mathsf{s}}$ are simply the diagonal elements $p(\mathsf{s})$. Therefore, using (15.4.23), which is repeated here

$$\mathcal{S}(\widehat{\rho}) = -\sum_i \lambda_i \log \lambda_i,$$

and setting λ_i equal to $p(\mathsf{s})$ gives

$$\begin{aligned}\mathcal{S}(\widehat{\rho}) &= -\sum_{\mathsf{s}} p(\mathsf{s}) \log p(\mathsf{s}) \\ &= H(\mathsf{s}),\end{aligned}$$

which is the Shannon entropy. When the states are not orthogonal, the sum $\widehat{\rho} = \sum_{\mathsf{s}} p(\mathsf{s})\widehat{\rho}_{\mathsf{s}}$ is not a diagonal matrix. For this case, the matrix has off-diagonal elements so that the eigenvalues of that matrix will not be equal to the diagonal elements of the matrix and will not be equal to $p(\mathsf{s})$. Accordingly, the von Neumann entropy will be less than the Shannon entropy, as state in (15.4.25).

16.9 *Von Neumann entropy*

A signal state is given by

$$|\psi\rangle = \tfrac{1}{2}|0\rangle + \tfrac{1}{2}\left(\cos\theta|0\rangle + \sin\theta|1\rangle\right)$$

(a) Determine the corresponding density matrix ρ.

Solution

View the given signal state as a statistical mixture of two pure states expressed in column form as

$$|\psi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\psi_1\rangle = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Forming the outer product $|\psi\rangle\langle\psi|$ for each signal state, the corresponding density matrix $\widehat{\rho}$ is

$$\begin{aligned}\widehat{\rho} &= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 1+\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.\end{aligned}$$

(b) Determine the eigenvalues of ρ.

Solution

The eigenvalues are the solutions to

$$\det\frac{1}{2}\begin{bmatrix} (1+\cos\theta)-\lambda & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta-\lambda \end{bmatrix} = 0.$$

with $\lambda_{1,2}$ given by

$$\lambda_{1,2} = \tfrac{1}{2}(1 \pm \cos\theta).$$

(c) Derive an expression for the von Neumann entropy as a function of θ.

Solution
Using (15.4.23) repeated here as

$$\mathcal{S}(\widehat{\rho}) \quad = \quad -\sum_i \lambda_i \log \lambda_i,$$

the von Neumann entropy is the Shannon entropy of the set of eigenvalues $\{\lambda_i\}$ with

$$\mathcal{S}(\widehat{\rho}(\theta)) \quad = \quad -\tfrac{1}{2}\left((1+\cos\theta)\left(\log(1+\cos\theta)-\log 2\right)+(1-\cos\theta)\left(\log(1-\cos\theta)-\log 2\right)\right).$$

(d) Determine the value of θ that maximizes the entropy.

Solution
Taking the derivative of the von Neumann entropy gives

$$\frac{\mathrm{d}\mathcal{S}(\widehat{\rho}(\theta))}{\mathrm{d}\theta} \quad = \quad \tfrac{1}{2}\sin\theta\log\left(\tfrac{1}{2}(\cos\theta+1)\right)-\tfrac{1}{2}\sin\theta\log\left(\tfrac{1}{2}(1-\cos\theta)\right).$$

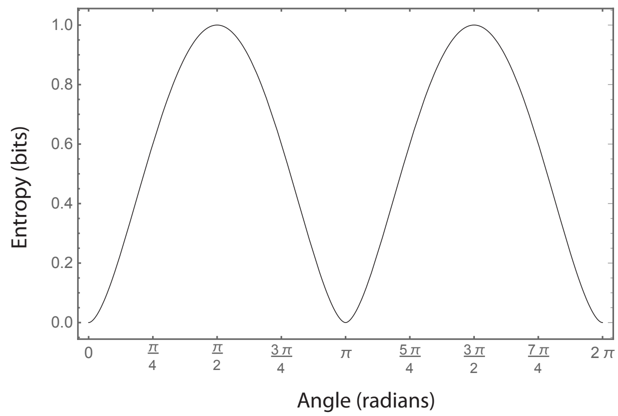
Setting this expression equal to zero and simplifying gives

$$\cos\theta \quad = \quad 0,$$

which has a solution $\theta = \pi/2 + n\pi$ where n is an integer. For this set of angles, the density matrix is an equiprobable mixture of two orthogonal signal states with the von Neumann entropy equal to the Shannon entropy.

(e) Plot the von Neumann entropy for $0 < \theta < 2\pi$ and demonstrate that the result determined in the previous step is correct.

Solution
A plot of the von Neumann entropy in bits as a function of θ is shown below.



The maxima for the range $[0, 2\pi)$ of angles shown on the figure occur at $\pi/2$ and $3\pi/2$ confirming the result derived in part (d).

16.10 *Small-signal expansion of the von Neumann entropy*

The von Neumann entropy of an antipodal coherent state is given by (16.6.5) and is repeated here

$$\begin{aligned}\mathcal{S}(\widehat{\rho}) &= H_b\big(\tfrac{1}{2}(1-e^{-2\mathsf{E}_\mathsf{b}})\big)\\ &= -(1-e^{-2\mathsf{E}_\mathsf{b}})\log(1-e^{-2\mathsf{E}_\mathsf{b}}) - (1+e^{-2\mathsf{E}_\mathsf{b}})\log(1+e^{-2\mathsf{E}_\mathsf{b}}).\end{aligned}$$

Using the power series expansion $\log(1+x) = x - \frac{x^2}{2} + O\left(x^3\right)$ and $e^x = 1 + x + \frac{x^2}{2} + O\left(x^3\right)$, show that the small-signal limit of the von Neumann entropy is given by

$$\mathcal{S}(\widehat{\rho}) \approx \mathsf{E}_\mathsf{b}\left(1 - \log \mathsf{E}_\mathsf{b}\right),$$

which is (16.6.7) and is the small-signal limit of the entropy of a Poisson probability mass function (cf. Problem 14.5).

Solution

The second expression in the problem statement is missing a factor of one half. Including this factor and using $e^{-x} \approx 1 - x$ gives

$$H_b\big(\tfrac{1}{2}(1-e^{-2\mathsf{E}_\mathsf{b}})\big) \approx (\mathsf{E}_\mathsf{b} - 1)\log(1-\mathsf{E}_\mathsf{b}) - \mathsf{E}_\mathsf{b}\log\mathsf{E}_\mathsf{b}.$$

Using $\log(1-x) \approx -x$ for the first term and discarding a term of order ${\mathsf{E}_\mathsf{b}}^2$ gives

$$\mathcal{S}(\widehat{\rho}) = H_b\big(\tfrac{1}{2}(1-e^{-2\mathsf{E}_\mathsf{b}})\big) \approx \mathsf{E}_\mathsf{b}\left(1 - \log\mathsf{E}_\mathsf{b}\right).$$

This expression shows that the small-signal limit of the von Neumann entropy for a antipodal coherent state approaches the entropy of a Poisson distribution.

16.12 *Codeword detection*

This problem works through the steps that derive the probability of a block-symbol-state detection error given in Section 16.4.3.

(a) Starting the Gram matrix $\mathbb{K}$ given in (16.4.9), derive the matrix $\mathbb{A}$ and its inverse $\mathbb{A}^{-1}$.

Solution

The Gram matrix $\mathbb{K}$ given in (16.4.9) has the following form

$$\mathbb{K} \quad = \quad \begin{bmatrix} 1 & \kappa_{\text{blk}} & \kappa_{\text{blk}} & \kappa_{\text{blk}} \\ \kappa_{\text{blk}} & 1 & \kappa_{\text{blk}} & \kappa_{\text{blk}} \\ \kappa_{\text{blk}} & \kappa_{\text{blk}} & 1 & \kappa_{\text{blk}} \\ \kappa_{\text{blk}} & \kappa_{\text{blk}} & \kappa_{\text{blk}} & 1 \end{bmatrix}.$$

The eigenvalues are the solution to $(\mathbb{K} - \lambda\mathbb{I})\mathbf{x} = 0$ where $\mathbf{x}$ is column vector. This system of equations has a nontrivial solution when the determinant of the left side is equal to zero which gives

$$\lambda^4 - 4\lambda^3 + 6\lambda^2\left(1 - \kappa_{\text{blk}}^2\right) - 4\lambda\left(2\kappa_{\text{blk}}^3 - 3\kappa_{\text{blk}}^2 + 1\right) - 3\kappa_{\text{blk}}^4 + 8\kappa_{\text{blk}}^3 - 6\kappa_{\text{blk}}^2 + 1 \quad = \quad 0.$$

This fourth-order polynomial factors into

$$(\lambda - \lambda_1)^3(\lambda - \lambda_2) \quad = \quad 0,$$

where $\lambda_1 = 1 - \kappa_{\text{blk}}$ and $\lambda_2 = 1 + 3\kappa_{\text{blk}}$. Therefore, the eigenvalues are $1 - \kappa_{\text{blk}}$, $1 - \kappa_{\text{blk}}$, $1 - \kappa_{\text{blk}}$ and $1 + 3\kappa_{\text{blk}}$. For the eigenvalue λ_1, the corresponding eigenvector $\mathbf{x}_1$ is any solution to the following set of homogeneous equations

$$\begin{array}{ccccccccc} (\mathbb{K}_{11} - \lambda_1)x_1 & + & \mathbb{K}_{12}x_2 & + & \mathbb{K}_{13}x_3 & + & \mathbb{K}_{14}x_4 & = & 0 \\ \mathbb{K}_{21}x_1 & + & \mathbb{K}_{22}x_2 & + & \mathbb{K}_{23}x_3 & + & \mathbb{K}_{24}x_4 & = & 0 \\ \mathbb{K}_{31}x_1 & + & \mathbb{K}_{32}x_2 & + & \mathbb{K}_{33}x_3 & + & \mathbb{K}_{34}x_4 & = & 0 \\ \mathbb{K}_{41}x_1 & + & \mathbb{K}_{42}x_2 & + & \mathbb{K}_{43}x_3 & + & \mathbb{K}_{44}x_4 & = & 0 \end{array}$$

One solution to this set of equations is $\mathbf{x}_1 = (0, 0, -1, 1)^T$ and is one eigenvector for the eigenvalue $1 - \kappa_{\text{blk}}$. Repeating this process gives two other eigenvectors $(0, -1, 0, 1)^T$ and $(-1, 0, 0, 1)^T$ for the eigenvalue $1 - \kappa_{\text{blk}}$ and $(1, 1, 1, 1)^T$ for eigenvector corresponding to $1 + 3\kappa_{\text{blk}}$. Organizing the column eigenvectors into a matrix gives

$$\mathbb{A} \quad = \quad \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The inverse of this matrix is

$$\mathbb{A}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

(b) Using these expressions, show that the matrix $\mathbb{M}$ can be written as

$$\mathbb{M} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix},$$

where $a = 3\sqrt{1-\kappa_{\text{blk}}} + \sqrt{3\kappa_{\text{blk}}+1}$ and $b = \sqrt{3\kappa_{\text{blk}}+1} - \sqrt{1-\kappa_{\text{blk}}}$. (Note the original problem statement had a typo of an additional square on each term containing κ_{blk}.)

Solution

The diagonal matrix $\mathbb{D}$ with diagonal elements that are the eigenvalues of the Gram matrix can be written as $\mathbb{D} = \mathbb{A}\mathbb{K}\mathbb{A}^{-1}$. Taking the square root of each eigenvalue, the matrix $\mathbb{M}$ can be written as

$$\begin{aligned} \mathbb{M} &= \mathbb{A}^{-1}\mathbb{D}^{1/2}\mathbb{A} \\ &= \frac{1}{4}\begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}, \end{aligned}$$

where $a = 3c + d$, $b = d - c$, $c = \sqrt{1-\kappa_{\text{blk}}}$, and $d = \sqrt{3\kappa_{\text{blk}}+1}$.

(c) Using the result from part (b), derive (16.4.10).

Solution

The conditional probability $p(k|\ell)$ is the squared magnitude $|m_{\ell k}|^2$ of each element of the matrix $\mathbb{M}$ (cf. (16.1.13)) with the squares of the on-diagonal elements $|m_{\ell\ell}|^2$ giving the probability of a correct decision $p(\ell|\ell)$. Because every diagonal element of $\mathbb{M}$ has the same form, $p(\ell|\ell) = \frac{1}{16}|a|^2 = \frac{1}{16}(3\sqrt{1-\kappa_{\text{blk}}} + \sqrt{3\kappa_{\text{blk}}+1})^2$, which is (16.4.10a). Similarly, every off-diagonal element has the same form with $p(\ell|\ell) = \frac{1}{16}|b|^2 = \frac{1}{16}(\sqrt{3\kappa_{\text{blk}}+1} - \sqrt{1-\kappa_{\text{blk}}})^2$, which is (16.4.10b).