Solutions to Exercises in Modern Condensed Matter Physics

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Note to Instructors

For a few of the more difficult problems, we include notes to the instructor suggesting simplifications, specializations and hints that the instructor may wish to give the students when assigning those problems.

Note also that there are some useful exercises within the appendices of the textbook.

Chapter 2

Ex. 2.1

(i)

The radiated electric field is

$$\vec{\epsilon}_a \approx r_{\rm e}\frac{e^{ikR_D}}{R_D}\left[\hat{n}\times(\hat{n}\times\vec{E}_{\rm in})\right]\; e^{-i\omega t}e^{-i\vec{q}\cdot\vec{r}}. \tag{1}$$

Replace $e^{-i\vec{q}\cdot\vec{r}}$ by $\langle e^{-i\vec{q}\cdot\vec{r}}\rangle = f(\vec{q})$. The vector $(\hat{n}\times\vec{E}_{\rm in})$ is perpendicular to both $\hat{n}$ and to $\vec{E}_{\rm in}$ and has length $|E_{\rm in}\sin\theta|$. Hence $|\hat{n}\times(\hat{n}\times\vec{E}_{\rm in})|^2 = E_{\rm in}^2\sin^2\theta$. Thus

$$|\epsilon_a|^2 = E_{\rm in}^2\frac{r_{\rm e}^2}{R_{\rm D}^2}\sin^2\theta|f(\vec{q})|^2. \tag{2}$$

The total radiated power passing through a sphere of radius $R_{\rm D}$ is

$$\begin{aligned} P &= cR_{\rm D}^2\int d\Omega\left(\frac{\epsilon_a^2}{8\pi}\times 2\right) && (3)\\ &= \frac{c}{4\pi}r_{\rm e}^2E_{\rm in}^2\int d\Omega\,\sin^2\theta|f(\vec{q})|^2. && (4)\end{aligned}$$

Let us normalize the incident electric field to that associated with a single photon in the normalization volume L^3

$$\frac{E_{\rm in}^2}{4\pi} = \frac{\hbar\omega}{L^3} = \frac{\hbar ck}{L^3} \tag{5}$$

which yields

$$P = \hbar c^2k\frac{r_{\rm e}^2}{L^3}\int d\Omega\,\sin^2\theta|f(\vec{q})|^2. \tag{6}$$

(ii)

Now compare this to the quantum result using the photon scattering matrix element in Eq. (2.28)

$$M = r_{\rm e}f(\vec{q})\wedge_k^2\hat{\epsilon}_{\vec{k}\lambda}\cdot\hat{\epsilon}_{\vec{k}'\lambda}. \tag{7}$$

Fermi's Golden Rule for the transition rate is

$$\Gamma = \frac{2\pi}{\hbar} \sum_{\lambda'} \frac{L^3}{(2\pi)^3} \int d^3k' \, r_{\rm e}^2 \wedge_k^4 [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{\epsilon}_{\vec{k}'\lambda}]^2 \delta(\hbar\omega - \hbar c k') |f(\vec{q})|^2. \tag{8}$$

Noting that the two polarization vectors $\hat{\epsilon}_{\vec{k}'\lambda'}$ and the vector $\hat{k}'$ are all mutually perpindicular, we find $\sum_{\lambda'} [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{\epsilon}_{\vec{k}'\lambda}]^2 = 1 - [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{k}']^2 = 1 - \cos^2\theta = \sin^2\theta$. The radiated power is

$$P = \hbar c^2 k \frac{r_{\rm e}^2}{L^3} \int d\Omega \, \sin^2\theta |f(\vec{q})|^2, \tag{9}$$

in agreement with the result from the semiclassical calculation.

Ex. 2.2

$$\begin{aligned}
S(\vec{q}) &= \frac{1}{N} < |W(\vec{q})|^2 >= \frac{1}{N} < \sum_{i=1}^{N} e^{i\vec{q}.\vec{r}_i} \sum_{j=1}^{N} e^{-i\vec{q}.\vec{r}_j} > && (10)\\
&= \frac{1}{N} < \sum_{i=j}^{N} e^{i\vec{q}.\vec{r}_i - i\vec{q}.\vec{r}_j} > + \frac{1}{N} < \sum_{i\neq j}^{N} \int d^3\vec{r} d^3\vec{r'} e^{i\vec{q}.\vec{r}_i - i\vec{q}.\vec{r}_j} \delta(\vec{r} - \vec{r}_i)\delta(\vec{r'} - \vec{r}_j) > && (11)\\
&= \frac{1}{N} N + \frac{1}{N} \int d^3\vec{r} d^3\vec{r'} e^{i\vec{q}\cdot(\vec{r} - \vec{r}')} < \sum_{i\neq j}^{N} \delta(\vec{r} - \vec{r}_i)\delta(\vec{r'} - \vec{r}_j) > && (12)
\end{aligned}$$

Remembering

$$< \sum_{i\neq j}^{N} \delta(\vec{r} - \vec{r}_i)\delta(\vec{r'} - \vec{r}_j) >= n^{(2)}(\vec{r'_i} - \vec{r}_j),$$

then obviously

$$S(\vec{q}) = 1 + \frac{1}{N} \int d^3\vec{r} d^3\vec{r'} e^{i\vec{q}\cdot(\vec{r} - \vec{r}')} n^{(2)}(\vec{r'} - \vec{r}) = 1 + n \int d^3\vec{r} e^{i\vec{q}.\vec{r}} g(\vec{r})$$

where we used $N/V = n$ and $n^{(2)}(\vec{R}) = n^2 g(\vec{R})$.

P.S. " <> " indicates thermal average in liquid or amorphous materials. It is unnecessary only for perfect lattices. Generally " <> " must be in the formula.